LONG TERM EFFECTS OF EXPOSURE TO PRIMARY HISTORICAL SOURCES IN UNDERGRADUATE STUDIES – THE CASE OF JULIUS

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> I've used the material later on. I've actually lend it out to others. I remember this; that we read these original texts, that we did these tasks. In a way, I actually think this is one of the things that I remember the clearest from my undergraduate mathematics courses. (Julius, 2015)

Abstract

The article addresses the question of which long-term effects an exposure to primary historical sources may have on students, exemplified by the case of Julius, who was exposed to extensive readings of such sources as an undergraduate student. The teaching module, in which the primary sources were a part, is described. The long term effect of the student is analyzed through a combination of two theoretical constructs; Niss' and Højgaard's overview and judgment and Barbin's special effects of using original sources in teaching mathematics. It is concluded that long-term effects are present in the case of Julius, and using the theoretical constructs it is described what these effects are.

Key words

Primary historical sources; original sources; guided reading; HAPh-modules; case study; long term effects.

Περίληψη

Το άρθρο θέτει το ερώτημα ποιες μακροπρόθεσμες επιδράσεις προκαλεί στους φοιτητές η επαφή τους με πρωτογενείς ιστορικές μαθηματικές πηγές. Ως παράδειγμα χρησιμοποιείται η περίπτωση, του Julius, που μελέτησε εκτενώς τέτοιες πηγές ως προπτυχιακός φοιτητής. Περιγράφεται το μάθημα του οποίου ήταν μέρος οι πρωτογενείς πηγές που μελέτησε. Στη συνέχεια αναλύονται οι μακροπρόθεσμες επιδράσεις στο φοιτητή με την χρήση δύο θεωρητικών κατασκευών: την λεγόμενη προσέγγιση «επισκόπησης και κρίσης» των Niss και Højgaard και την θεώρηση της Barbin που αφορά στις ειδικές επιδράσεις της χρήσης πρωτότυπων πηγών στη διδασκαλία των Μαθηματικών. Η ανάλυση επιβεβαιώνει την ύπαρξη μακροπρόθεσμων επιδράσεων στη περίπτωση του Julius, οι οποίες περιγράφονται με την χρήση των προαναφερόμενων θεωρητικών κατασκευών.

Λέξεις Κλειδιά

Πρωτογενείς ιστορικές πηγές, αυθεντικές πηγές, καθοδηγούμενη μελέτη, HAPh-μαθήματα, μελέτη περίπτωσης, μακροπρόθεσμες επιδράσεις.

0. Introduction

All of us, who have used elements of history of mathematics with our mathematics students in a more or less extensive way, have probably wondered what the students gained in the long run from this exposure to history, what aspects stayed with them, and how they in hindsight view their earlier encounter with the history of mathematics. Of course, when students have completed their studies, they often disappear out of one's periphery. For that reason, I decided to catch one of my old students – *Julius* – in the spring term of 2015, just before he was about to complete his university studies, and interview him about his experiences and hindsight reflections concerning a historical project embedded in an undergraduate discrete mathematics course, I gave at Roskilde University in the spring term of 2012. But allow me to provide a bit more of the background story on this particular historical project and also on how the paths of this particular student and myself have crossed each other, and how the idea for this particular article came about.

In 2010 and 2011 I did a postdoc at the University of Southern Denmark, one purpose of which was to look further into the use of history of mathematics in Danish upper secondary schools. In particular, I attempted to design a couple of teaching modules which addressed *H*istory of mathematics, *A*pplication of mathematics, and aspects of *Ph*ilosophy of mathematics in unison – modules later referred to as *HAPh*-modules (e.g. Jankvist, 2012; 2013; 2014a; 2014b). Both of these modules were implemented in an upper secondary mathematics class, and eventually published as teaching materials in Danish.[1] At the beginning of 2012, I was employed at Roskilde University, and asked to co-teach a course in discrete mathematics, as part of the bachelor science program, a course which I had also taught there back when I was a doctoral student. This course is a more or less classic introductory course to discrete mathematics, involving topics such as propositional logic, basic set theory, algorithms and their complexity, recursivity, elementary number theory, basics of counting, permutations, combinations, relations, etc. At the end of this course, the students were to do a couple of mini-project assignments. One of these concerned logic programming, and was given by co-teacher, a mathematical computer scientist. The other project was left for me to decide; and having already tried out the two HAPh-modules with upper secondary students, I was eager to try them out also with first and second year university students.

As mentioned there were two HAPh-modules: one on graph theory and one on Boolean algebra. The students of the class were asked to choose one or the other, and to work through the modules in groups of approximately four. The student, Julius, who makes up the case study for this article was in a group who did the module on graph theory. Julius later went on to become a student of both mathematics and of history at Roskilde University. In the spring of 2013, I was employed at Aarhus University's Campus in Emdrup, Copenhagen, and was quite surprised to find Julius sitting in the reception. It turned out that he had been working there as a student helper since he was in upper secondary school. This meant that for a couple of years have been able to follow Julius quite closely, often having discussions with him regarding his studies, etc. For that reason, I decided to arrange a couple of interview sessions with him before he finally graduates, and 'disappears'.

In the following, I shall briefly describe the basic idea and design principles the HAPh-modules, and present the module on graph theory, since this module will make up the basis for the interviews with the student. (For a description of the other module, see Jankvist, 2013; 2014b). As part of this description, I shall spend some time reflecting on the essay assignment that the students were given as part of the module, and provide excerpts from their hand-in mini-project. Next, the interviews with the student Julius is presented and discussed. Finally, some potential conclusions are discussed in the relation to the questions which initially spurred on the idea for the article.

1. Designing a HAPh-module

The main idea for the HAPh-modules was spurred on by the Danish KOM-project's articulation of three kinds of so-called overview and judgment, which are "'active insights' into the nature and role of mathematics in the world" the purpose of which is to "enable the person mastering them to have a set of views allowing him or her *overview and judgement of the relations between mathematics and in conditions and chances in nature, society and culture*" (Niss & H jgaard, 2011, pp. 49, 73, italics in original). The three types of overview and judgment (OJ) are: (OJ1) the actual application of mathematics in other subject and practice areas; (OJ2) the historical evolution of mathematics, both internally and from a social point of view; and (OJ3) the nature of mathematics as a subject. As argued in Jankvist (2013), these three kinds of overview and judgment may be brought about by focusing on aspects, or exemplary cases (Jankvist, 2011), from the history (OJ2), the application (OJ1), and the philosophy (OJ3) of mathematics. Of course, the three types of overview and judgment may be seen as to concern meta-issues of mathematics (Jankvist, 2009), but as Niss and H jgaard (2011) point out, it is clear that if overview and judgment regarding the historical evolution of mathematics (OJ2) is to have any *weight* or *solidness*, it must rest on *concrete examples* from the history of mathematics – or to put it differently, reflections concerning the meta-issues must somehow build on or be anchored in concrete mathematical in-issues (Jankvist, 2011). Obviously, a similar argument holds for the other types of overview and judgment.

To realize the above idea, it was decided to have one overall theme for each of the HAPh-modules, to have the students read one primary historical source representing the historical, applicational, and philosophical aspects, respectively, and to end the module with an essay assignment. The reason for 'resorting' to primary historical, or original, sources has to do with what Barbin (1997) and Jahnke et al. (2000)[ii] refer to as the three general ideas best suited for describing the special effects of using original sources in mathematics education: *replacement* (or in the original French wording; *fonction vicariante*), which refers to the replacement of the usual with something different, for example by allowing mathematics to be seen as more than just a corpus of knowledge and techniques; *estrangement* or *reorientation* (*fonction dépaysante*, or *dépaysement*), which challenges one's perception by making the familiar unfamiliar, thus also causing a reorientation of our views, and; *cultural understanding* (*fonction culturelle*), which allows us to place the development of mathematics in a scientific, technological, or societal context of a given time and place. However, because primary historical sources may often be difficult to access, the presentation of these in the modules were supplied with explanatory comments and illustrative tasks along the way. This form of presentation and way of working is referred to as a *guided reading* of primary original sources, a described by Barnett, Lodder and Pengelly (2014). Practically no mathematical requirements were needed beforehand on the students' behalf to study the text of Euler – a major reason for choosing this text initially – and many of those needed for the Dijkstra text were introduced in the guiding commentaries along with the Euler text, thereby also bringing the students somewhat up to date with modern notation, terminology, etc.

As for the essay assignments, I have previously found that this is a good way of bringing small groups of students to work with meta-issues of mathematics (Jankvist, 2011). The particular setting creates a scene, where students at the end of the implementation of the teaching module, after having read and worked with the mathematical case in question, are to discuss among themselves meta-issues regarding the case. These meta-issues are chosen beforehand and included in the description of the essay assignment at the end of the teaching material; although in such a manner that the students can draw in additional meta-issues should they find it relevant. In the module on graph theory, the students first and foremost were to relate the two first texts on history and applications to the philosophical discussion in the third text as part of their essay assignments – a task which to some extent also forces out some of the interplay between the three dimensions, although still exemplified by the concrete case of graph theory and the chosen overall theme. In the following section, I describe this overall theme and briefly introduce the three texts.

2. A module on Euler paths, shortest path, and minimum spanning trees

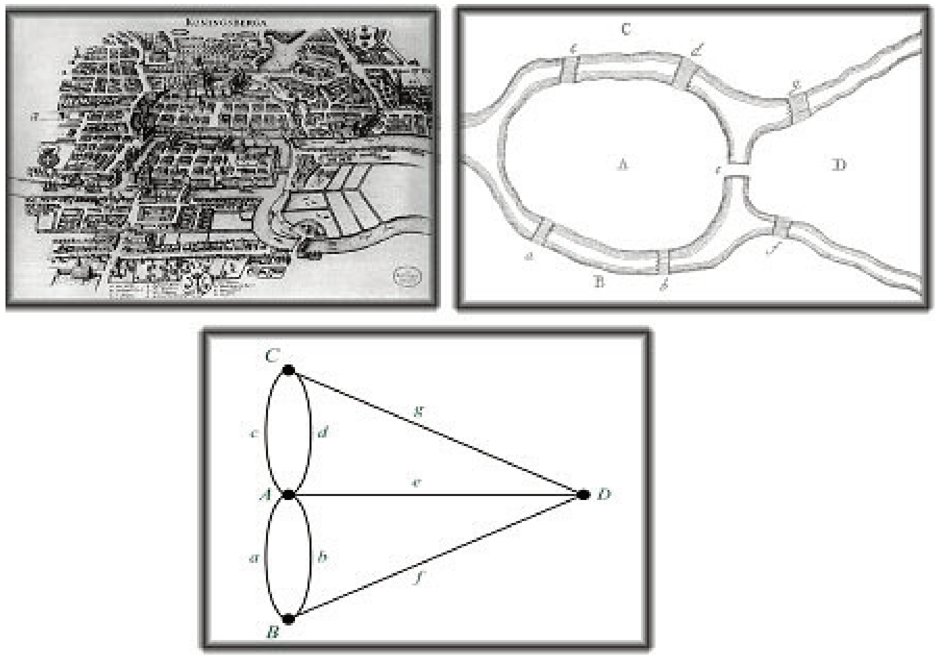
The three texts (in Danish translation) included in the teaching material for the module on graph theory were:

- Leonhard Euler, 1736: *Solutio problematis ad geometriam situs pertinentis*
- Edsger W. Dijkstra, 1959: *A Note on Two Problems in Connexion with Graphs*
- David Hilbert, 1900: *Mathematische Probleme – Vortrag, gehalten auf dem internationalen Mathematiker-Kongreß zu Paris 1900* (the introduction only).

The module was conceived as to have an overall theme, this being *mathematical problems*, which was what Hilbert addressed in general terms in the introduction of his lecture from 1900. To make Hilbert's general observations somewhat more concrete, the students were first to read the two other texts, each of which addresses a mathematical problem. Euler's paper from 1736 is on the *Königsberg bridge problem*: how to take a stroll through Königsberg crossing each of its seven bridges once and only once – and today the paper is considered the beginning of mathematical graph theory. Two centuries later, with the dawn of the computer era, graph theory (and discrete mathematics in general) found new applications. *Dijkstra's algorithm* from 1959 solves the problem of finding shortest path in a connected and weighted graph, and today it finds its use in almost every Internet application that has to do with shortest distance, fastest distance or lowest cost. Furthermore Dijkstra also discussed a method for finding *minimum spanning trees*, a problem relevant for the building of computers at the time, but also highly relevant today.

The students' way into the first original text was by looking at Euler's diagram of landmasses and rivers in Königsberg (figure 2, middle) and then verify that this is in fact an accurate representation (or model) of the Königsberg bridge problem by comparing with an illustration of the town (figure 2, left). Afterwards the students were told that in modern graph theory, landmasses are represented by vertices (or nodes) and links between them by edges. Students were asked to transform Euler's diagram into such a modern graph individually and then compare their own representation to their fellow classmates, this illustrating that graph representations can look different. The idea was to have the students adapt more and more schematic representations of the Königsberg bridge problem until arriving at something looking like figure 2 (right), gradually increasing the level of abstractness.

Figure 2. Left: An illustration of Königsberg with its 7 bridges from 1652
Right: Euler's 1736 simplification of Königsberg's bridges
Bottom: A modern graph representation



Once being familiar with the modern representation of a graph, the students were introduced to the problem of representing *multiple edges*, such as for example the two edges between vertices *A* and *B* in the Königsberg graph. These cannot be represented by only their pair, (*A*,*B*), since this causes ambiguity (which is why Euler also named them *a* and *b*, respectively). To illustrate a formal and general way of dealing with this to the students, they were provided with the following modern definition:

> A graph ***G*** is a set of vertices *V*(***G***) and a set of edges *E*(***G***) together with a function *ψ*, which for every edge e *E*(***G***) assigns a pair, called *ψ*(e), of vertices from *V*(***G***).

The students were then asked to write up the sets *V*(***G***) and *E*(***G***) for the Königsberg graph and the seven function values of *ψ*(e). On the one hand, the idea of this was to enable them to perceive the definition of a graph as a triplet ***G***={*V*(***G***), *E*(***G***), $\psi_{\boldsymbol{G}}$}, and on the other hand to have them realize how the above definition in a general fashion resolves the problem of ambiguity, when two vertices in a graph have multiple edges.

As Euler, in his text, introduces various constructs, the students were introduced to the somewhat equivalent modern terminology in the intermediate commentaries, e.g. *route*, *path*, *Euler path* (open and closed), *subgraph*, *degree* of a vertex as well as a few small theorems which Euler explicitly or implicitly uses, such as for example the so-called *handshake theorem*. At the end of his paper, Euler states his three main results (Euler, 1736, p. 139 in Fleischner, 1990, p. II.19, numbering is mine):

(i) If there are more than two regions with an odd number of bridges leading to them, it can be declared with certainty that such a walk is impossible.

(ii) If, however, there are only two regions with an odd number of bridges leading to them, a walk is possible provided the walk starts in one of these two regions.

(iii) If, finally, there is no region at all with an odd number of bridges leading to it, a walk in the desired manner is possible and can begin in any region.

The students were first asked to formulate these three results using the modern terminology and notation they had been introduced to. Next, they were provided with a modern definition of a *connected graph*, i.e. that there exists a route between every pair of vertices, a property Euler does not state explicitly. Using this property, the three results may be reformulated as:

> If a connected graph $\boldsymbol{G}$ has more than two vertices of uneven degree, then it does not contain an Euler path.
>
> Let $\boldsymbol{G}$ be a connected graph, then $\boldsymbol{G}$ contains an (open) Euler path if and only if $\boldsymbol{G}$ contains exactly two vertices of uneven degree.
>
> Let $\boldsymbol{G}$ be a connected graph, then $\boldsymbol{G}$ contains a (closed) Euler path if and only if all vertices of $\boldsymbol{G}$ have even degree.

Most of Euler's efforts goes into proving his first result (i), and regarding the third (iii), which today is considered the main theorem of the paper, he only proves it in one direction. To introduce the students to the notion of if-and-only-if theorems, they were to consider result i as being of the form $\mathbf{P} : \mathbf{A} \Rightarrow \mathbf{B}$, and then identify $\mathbf{P}$, $\mathbf{A}$, and $\mathbf{B}$. After having the students prove that $\mathbf{A} \Rightarrow \mathbf{B} \equiv \neg\mathbf{A} \Leftarrow \neg\,\mathbf{B}$ (by means of a truth table), they were asked to write up $\neg\mathbf{B} \Rightarrow \neg\mathbf{A}$ for result i, i.e. formulating the contrapositive of this theorem, which states that

> If $\boldsymbol{G}$ is connected and has an Euler path (open or closed), then $\boldsymbol{G}$ has two or less vertices of uneven degree.

Since Euler has shown, in his own context of course, that a graph will always contain an even number of vertices with uneven degree, we may distinguish between two different cases: when $\boldsymbol{G}$ has exactly two vertices of uneven degree and when it has none, i.e. when all vertices have even degree. These cases correspond to the $\Rightarrow$–direction in results ii and iii, respectively. Thus, by looking at Euler's original text again, the students would be able to deduce that the missing parts of the proofs are the $\Leftarrow$ –directions for results ii and iii. For result iii this is ascribed to Carl Hierholzer (published posthumous in 1873), and the students were shown this proof. Then they were asked to prove the $\Rightarrow$–direction for iii and both ways for result ii using modern terminology.

While employed at *Mathematical Centrum* in Amsterdam in 1956, Dijkstra was asked to demonstrate how powerful the center's computer, the so-called ARMAC, was. He did so by devising an algorithm for finding shortest path between two nodes in a connected, weighted graph – today known simply as *Dijkstra's algorithm*. Dijkstra's description of his algorithm appeared in 1959 in a paper which also describes an algorithm for finding minimum spanning trees in connected, weighted graphs. Unlike Euler's text the text by Dijkstra is short and builds on a large apparatus of existing graph theory. In fact, the text is only a few pages long. Also, Dijkstra only provides the description of his algorithms and he gives no examples of running these and no proofs of their correctness either, only a few remarks about running time. Thus, this text needed some 'unpacking' for the students in the form of explanatory comments, additional examples, tasks, etc. For example, the students were provided with definitions of a *weighted graph*, a *tree*, and a *spanning tree*:

> A connected graph $\boldsymbol{T}$ without any subgraphs that are circuits is called a tree, and a tree that for some graph $\boldsymbol{G}$ contains all vertices of $V(\boldsymbol{G})$ is called a spanning tree.

To illustrate that finding a least spanning tree is not trivial, the students were asked to look at the Königsberg graph (figure 2, right) and find the number of different spanning trees that can be constructed from this, and then explain their method for finding the answer. (The answer, which is 21, may be calculated using the so-called (Kirchhoff-Trent) *Matrix-Gerüst-Satz*. Deleting the *i*'th row and column of this matrix and taking the determinant of the one dimension smaller matrix reveals it. But the students had to do it by systematic inspection.)

In fact, Dijkstra's motivation for devising an algorithm for finding minimum spanning tree had to do with a very specific problem related to the construction of the ARMAC computer. The massive size computers at the time required vast amounts of expensive copper wire to connect their components. Finding a minimum spanning tree corresponds to leading electricity to all electric circuits while using the least amount of expensive copper wire. (A few comments were of course made to the students about the earlier discoveries of the algorithms by Jarn k, Bor kva, Kruskal and Prim, respectively.)

Having worked through the Dijkstra text, the commentaries and examples to this, and a modern proof of the shortest path algorithm's correctness, the students got to the third text by Hilbert; the introduction of his 1900-lecture in which he discusses 'mathematical problems'. Paraphrasing Hilbert roughly, he states that often some mathematical development is spurred on by a problem in the extra-mathematical world. Then it is drawn into mathematics and rephrased so that it is hardly recognizable anymore and embedded in a much more general context. Years later, when this has grown into a mathematical discipline, what often happens is that it may then again be used to solve some new extra-mathematical problem:

> Surely the first and oldest problems in every branch of mathematics spring from experience and are suggested by the world of external phenomena. [...]
>
> But, in the further development of a branch of mathematics, the human mind, encouraged by the success of its solutions, becomes conscious of its independence. It evolves from itself alone, often without appreciable influence from without, by means of logical combination, generalization, specialization, by separating and collecting ideas in fortunate ways, new and fruitful problems, and appears then itself as the real questioner. [...]
>
> In the meantime, while the creative power of pure reason is at work, the outer world again comes into play, forces upon us new questions from actual experience, opens up new branches of mathematics, and while we seek to conquer these new fields of knowledge for the realm of pure thought, we often find the answers to old unsolved problems and thus at the same time advance most successfully the old theories. (Hilbert, 1902, quoted from the 2000-reprint, p. 409)

In a certain sense, the case of graph theory illustrates this: first, spurred on by the Königsberg bridge problem, which Euler generalized so that the answer to the original problem falls out as a small corollary to his more general results; and next, two centuries later when we have a much clearer idea about the discipline of graph theory, Dijkstra solves the extra-mathematical problem of shortest path (and also considers minimum spanning trees) in this graph theoretical context.

3. Three essay assignments

For the students to realize the above connection between the three original texts, and thus the three dimensions of history, application, and philosophy, they were asked to identify the criteria that Hilbert proposed for a good mathematical problem (e.g. that it must be explainable to laymen and that it must be challenging but not inaccessible, etc.) and see to what degree the problems treated by Euler and Dijkstra fulfill these, and then relate them to Hilbert's comments on the development of mathematics in general. The module included three essay assignments, each addressing different aspects in relation to overview and judgment.

The first essay was on the just discussed topic of *mathematical problems*, linking the three texts by Euler, Dijkstra, and Hilbert together. In their essay, Julius' group writes the following:

> The requirement that everyone in principle should be able to understand the problem is fulfilled by the problems [in the texts], since they do not require any mathematical specialist knowledge, but on the contrary concerns contemplations regarding general issues. [...]
>
> Concerning the development of mathematics [as a discipline], Hilbert [1902, p. 437] says: "History teaches the continuity of the development of science. We know that every age has its own problems, which the following age either solves or casts aside as profitless and replaces by new ones." The case in question fits nicely with this description by Hilbert, since when Euler tried to describe the [Königsberg bridge] problem, he began with the concrete issue and developed a general description of this by means of assigning symbols, etc. Later on a further generalization took place by introducing vertices and edges, this generalization then being used to solve yet other, though related, problems, as for example that of shortest path. (Julius' group, March, 2012)

The second essay was on *mathematical proofs* and first dealt with different kinds of proofs and proof techniques as well as the use and need for new signs and symbols (both arithmetical and graphical) in the development of new mathematics (concepts, definitions, etc.), something that Hilbert also addresses. The students were asked to discuss this with relation to Hilbert's text and try to draw connections to the two cases, in particular the advantages Dijkstra had in 1959 with a fully developed graph theoretical and conceptual apparatus at his disposal, as compared to Euler who had to start from scratch in 1736. Finally, the students were asked to look into Hilbert's actual discussion of proofs and their role in solving mathematical problems as well as the role of rigor in mathematical proofs. On the overall, the idea of this was to spur some reflections on the students' behalf regarding the epistemological development of the notion of proof. This final part is not really well developed in the essay by Julius' group, but they offer some reflections on the question of the 'graph theoretical and conceptual apparatus':

> These days one has a lager selection of well-defined notations and concepts at one's disposal when having to put forward mathematical theorems and formulate proofs. We may notice the difference if we look at the length of modern formulations of old proofs. The modern proofs are shorter and appear more concise in their formulation. [...]
>
> Since Euler was the first to treat a [graph theoretical] problem, he did not have a lot of mathematical tools to choose between. Instead he had to define the necessary concepts himself. When Dijkstra wrote his article, he could draw on the already developed definitions in graph theory, and from mathematics in general. (Julius' group, March, 2012).

The third essay was about *mathematics' status as a (scientific) discipline*, in its own right and in comparison to other disciplines, e.g. physics. Based on their readings of Hilbert, and the two texts by Euler and Dijkstra, the students were asked to try to point out some characteristics of mathematical problems, methods, and ways of thinking as well as to say something about the types of results mathematics delivers and what they may possibly be used for. They were invited to discuss this by comparing mathematics to other academic disciplines. Then they were asked to identify what Hilbert said about the differences and connections between mathematics and other disciplines, and then discuss to what extent they agree or disagree. With reference to their second essay, Julius' group point out the following:

> Mathematics has proofs, which other disciplines do not – proofs are not possible in the same manner in other disciplines, one has to document by means of experiments. In mathematics, one concerns oneself with abstracts entities within abstract structures, which Hilbert also talks about. Also, Hilbert discusses how mathematical knowledge can be applied within other subject areas. [...] In short, mathematics delivers solutions to general problems [outside of mathematics].
>
> Hilbert describes the difference between mathematics and science-related disciplines. [Unlike these] he sees mathematics as kind of unity, which no good will come of splitting up into smaller braches, because the different branches then cannot benefit from the results of one another. (Julius' group, March, 2012)

Finally, as the very last questions of the essay assignment, students were asked to comment on their experiences with having to read the three original sources. Julius' group said:

> We all agree that Euler's text was the best, because you take a real-life problem and transform it to a mathematical problem, which makes it easily understandable and entertaining at the same time. [Dijkstra's] text was compact, which made it hard-to-reach at first encounter, although both the approach and results were quite clear. [...] Since Hilbert's questions have contributed to shaping which mathematical problems prosperity has been concerned with, it is interesting to see what his personal views on mathematics were. (Julius' group, March, 2012)

4. Interviews with Julius

In February of 2015, Julius was in the fifth year of his university studies at Roskilde University, 23 years of age, and doing a combined master's degree in mathematics and history. In this respect, Julius was of course a bit special since he majors in subjects that are topics in the HAPh-modules. But as we shall see in the interview excerpts to follow, some of which are rather lengthy, this had some interesting consequences for his assessment of and hindsight reflections about the module and the exposure to historical source material. Furthermore, Julius had in mind to become an upper secondary school teacher, which resulted in a few didactic and pedagogical considerations as well, both concerning the module and mathematics teaching in general.

I conducted two interviews with Julius; a first somewhat long interview (45 minutes), and a second shorter follow-up interview (25 minutes). The interviews were had the form of collegial conversations and were semi-structured in the sense that although I had prepared an interview guide in advance, we would deviate from this time and again, if other interesting aspects aroused as part of the conversation. The following presentation of the interviews follows the conversation with Julius, i.e. the excerpts appear in the order which they occurred during the interview. The very first utterance of Julius was that which appears in the beginning of this article, and which was his reply to whether or not he recalled the module. Following this, Julius continued:

> I remember the Königsberg bridge problem clearly. It is a problem that I know in details. We were dragged through this – or guided through it, in a pedagogical way, right, while at the same time reading in the original about these issues. I'd definitely say that this module is one of the things I remember the best... What struck me at the time, and still does, and what impressed me, was that a single problem... I mean, how unimportant it is in principle, if you can walk through a town in a certain way... But then you generalize it, take it out of the town, out of its reality with bridges and islands. [...] I thought that was pretty impressive. That every time he [Euler] had a problem, he would see that you could write it up in a more general way, express it in a new fashion, so he could realize something new about it. The insight that this is what you often do in mathematics; try to generalize it as much as possible to find the underlying system, I mean take the problem out of its real context – that was exciting, seeing this being unfolded through the texts. Or through the original texts – as original as they can be – and also by way of getting some tasks, where instead of just being told, you had to realize it for yourself. [...] There were several tasks of the kind where the idea of you doing the task was to make you realize what Euler realized at this particular place of his text. At least that is how I perceived some of the tasks. [...] So we weren't just being told what was interesting. You had to discover some of the interesting points yourself. Often in math it's like, okay, here is a proof, here is a theorem, here are some examples, and some tasks for you to apply the theorem on. Whereas here you got a different understanding of what was going on, mathematically speaking, than the classical 'proof-theorem-kind-of-way'. Here you obtained realization, which meant that you could follow what was being talked about. (Julius, February 18, 2015)

Julius' answer above referring to the explanatory tasks led me to ask him another question from the interview guide, namely what he thought of the guiding reading approach:

> You challenge the classical way in which you learn something; because it's clear that some of the definitions and concepts are still there. But overall, the tasks were not designed for you to test something [i.e. a theorem] that you had to learn. You were not supposed to learn something specific by doing the task. You were to get some insight to help you better understand the original source. So, I think, the connection between the historical and the mathematical is much more substantial in this work. And you obtained an understanding of how Euler had thought – okay, not one hundred percent, of course – but you got the feeling that you did, that you could follow what went on in the original text. I mean, it's a different way of

> viewing a proof. And this was fun seeing unfolded at the time. [...] Of course, it is a major leap between the two texts [Euler's and Dijkstra's]. But still, there is a clear connection between them. It is fun to see how a single question can become almost an entire theory, after years of research, right, by different people. That was impressive. [...] It is fairly obvious, how it can be relevant and exciting to read. And that you see how the notation has changed. It is interesting to find out how this mathematics has come into being, right. I mean, if you are used to always being presented to mathematics kind of top down, with modern definitions, etc., then it is completely strange being thrown into an old text like this. Personally, I was already interested in mathematics and history, but this made my interest even bigger, or how should I put it... I mean, seeing it unfold in this way, seeing how you can cut up a text and present it so also people less nerdy than me would find it interesting to read. (Julius, February 18, 2015)

From the conversation with Julius, it was clear that he found that he from the module had learnt some history of mathematics. A natural question to ask was if he also found that he had learned some mathematics?

> I believe I did. [...] For example, doing these graphs [points to a figure similar to figure 1]. I clearly remember the shortest path problem [Dijkstra]. I made some projects later, where we did similar things, weighted edges and stuff. [...] But what I benefitted the most from, or at least what I feel I benefitted the most from, was that about the 'nature of mathematics'. I mean, also when you tell other people about mathematics, and when you think about how it actually develops; you look at something and then you generalize it. Then other people catch on and do something with it. I mean, it is not a straight path of someone initially thinking 'now I want to develop a theory that can be applied for exactly this thing'. It is rarely like that – at least in what I have seen: here someone looks at something specific and formulate a problem, which other people then try to solve for some 300 years. All of sudden you then realize that the methods used were not applicable at all for this specific problem, but that they could solve another different one. So often it is a rather rough and uneven path, and a kind of crooked way in which things are connected, and one which is very difficult to predict. [...]
>
> In the textbooks things are presented in a straight way – and as if they came into being in the same manner. So just obtaining an understanding of that often it is actually very, very different was exciting – also to see it for yourself. You don't think about it when reading you usual textbook, because the presentation is so top-down and stripped from all historical context. Also if you have a lecturer who just does things, and you are like... I mean, it is not obvious how much thought people had to do for decades or even centuries before arriving at anything looking like what you are being presented to on the blackboard during a single lesson, right. So you got an understanding of how mathematics has developed, how it came into being, and what make people throw them self at a mathematical problem. This became clear to me. And I don't think it was something I had really thought about in the same way before, not until we saw the sources and what people did. [...] There are some questions I wouldn't have posed, some thoughts

> about mathematics which I wouldn't have made, had I not seen it being unfolded in this particular way. (Julius, February 18, 2015)

Referring back to the essay assignment on the nature of mathematics as a discipline, I wanted to ask Julius on his current views on this, i.e. if and how mathematics is different from other (scientific) disciplines – also to see if his hindsight reflections were somehow different from the answer of his group in 2012.

> If you are to compare how Euler worked to some of the other disciplines, then it is obvious that it is something different. If a physicist is to solve a problem, he would work within physics; he would use some theories and tools from physics. But he would always return to the outset, return to the original problem. But in the Euler text there is almost no connection facing backwards – only forward. He generalizes, generalizes, generalizes, and fairly soon he couldn't care less about the Königsberg problem. He quickly realizes that he can answer this. But he doesn't answer it by actually answering the Königsberg problem. He answers it by being capable of answering a long line of similar questions; his answer is a "theory". He won't accept to solve just the one problem. He could have done that, of course. I mean spent three pages on solving only this one problem. But if he had *only* solved the Königsberg problem, then we would hardly ever have heard about it afterwards. And this I think is typical for mathematics. You generalize and take things further and further away from the outset. And this you get an impression of – or you get an image of how mathematics is as a discipline. (Julius, February 18, 2015)

Throughout both interviews, Julius keeps returning to two overall issues. The first is the above mentioned one of Euler's generalization of the Königsberg bridge problem to something which in essence has very little to do with neither Königsberg nor bridges. The other issue concerns the fact that mathematics developed in one context may later be applied in different contexts – one of the points made by Hilbert. Julius personally talked about the "external" and the "internal" influences on the development of mathematics – what is usually also referred to as inner and outer driving forces (Jankvist, 2011).

> Actually, it is a good example of exactly that, i.e. what began as bridges in a town ended up being used for something with telegraph cables. And then you realized – because you had the general theory – that telegraph cables and cobber wire connecting components in computer hardware, things which at first sight are not related in any way, actually is the same problem, only in different forms, because you realize that it can be treated as the same mathematical problem...
>
> Yes, the minimum spanning tree.
>
> Yes! Exactly, right.
>
> And you remember this after three years?
>
> Yes, yes. I remember we talked about this. I thought it was funny because of these problems, which one wouldn't immediately think to be related; I mean, if I list ten

> things and someone had to say which of these were related, or could be solved in the same mathematical way. [...] It has to do with how time changes. At one point in time you need telegraph cables, later you develop the computer, and you realize it involves a similar problem, only in a new form. [...] It has to do with the external world, how it develops, and how mathematics is brought into play. It is fun to see that it begins with an external question, which is dragged into mathematics, and then develops within the field of mathematics – internally, right – [...] and then it plays back at a problem that is not related to the initial problem. So, this thing with the telegraph cables, I remember very clearly. I mean that the mathematics can be used for different things which are not related. This, and then the thing with you generalizing a problem more and more, which I would say is part of the nature of mathematics; that you look for the simplest expression of something, the simplest problem of its kind, or the simplest way of writing up a whole line of problems – this is what you often want to do, right. You don't want to talk about just one problem. You want to say something about all problems having this particular form. (Julius, February 18, 2015)

Next, I asked Julius whether the Hilbert text had left any impression with him. At first sight apparently not, since he did not recall much of it. For that reason we agreed that he would reread the Hilbert text and we would talk briefly about this a week later. Maybe he would recall aspects of it, when rereading it. Also, there might be other hindsight reflections appearing from a rereading of the text that could be of interest for this particular article.

In the second interview, Julius again talked about things which he and his group had discussed in the essay assignments back in 2012, e.g. Hilbert's definition of a good mathematical problem and why both the Königsberg bridge problem and the shortest path problem fulfilled Hilbert's criteria. But one interesting thing which Julius brought up was related to the essay referring to Hilbert seeing no meaning in splitting mathematics into several braches, as is done in the natural sciences. Julius found that he could relate to what Hilbert said, also from doing various project works as part of his education at Roskilde University. But what he also pointed out was when being taught mathematics in regular courses, it often did appear as being divided into different branches – or "boxes" – which did not necessarily appear to be related.

> When you are being taught math, then you have analysis, and maybe it is called 'fourth dimension analysis' and you get the impression that it makes sense to have this division of topics into boxes. But when we do our student projects [at Roskilde University], then it isn't always clear what math to use... I mean, 'what do you have to say something about here?' and then you go for what you know. Then it is nice to see how Euler did. He wasn't restricted to a specific box to begin with. He looked at a problem, and then he thought about which things he needed to describe it. He wasn't in a predefined box. He chose elements from different branches of mathematics that he needed to describe his problem. So the texts by Euler and Dijkstra and their ways of working with mathematics illustrate well that

even though you may have been taught mathematics as being divided into boxes, when you actually have to use it for something, then it is not the case that if you are within this box, then you can only use the elements of this box. You use what makes sense. And if you can provide arguments, and if it makes sense within the system, then you can make use of the things you need. And that I think you get an idea about from these texts. (Julius, February 26, 2015)

Another thing which Julius brought up, and which was a bit surprising for me, was Gödel's incompleteness theorems. In the material surrounding the three primary historical sources, I took the opportunity to tell the story of Gödel's incompleteness theorems, since the first of these was presented in Königsberg on September 7, 1930 – the day before Hilbert gave a famous radio speech on mathematics, in which he said his immortal words: "Wir müssen wissen. Wir werden wissen." ("We must know. We will know").[iii] In the material, the two theorems and their consequences were outlined briefly, and connected to the history of Hilbert's speech, and the reactions from the mathematical community to Gödel's theorems.

Gödel's incompleteness theorems, right – I mean, I've always had an idea that you could draw a circle around mathematics, and say that this you know about mathematics, right. You've always had an idea that you were on rock solid ground, and that you could build from there. The idea that something at the bottom is not solid – you must give up on this. This has had a major influence on how I understand the discipline I work within – because you don't have this foundation. You can create temporary foundations, but you just have to accept, that there are things which you can't know for certain – or how you want to put it. I thought that was strange. I still think it is a little strange.

And you remember this? You remember Gödel's theorems from the course?

Gödel's theorems! Yes! I remember them from back then. And when I reread the material I recalled it. It was so strange, I thought. Of course, it wasn't like with the mathematicians described here, it didn't shake me to the core. I wasn't shocked like that, since I was still in the process of learning the subject. But still I was like; for real, is that how it is! Because in upper secondary school things were presented as this-is-how-it-is kind of knowledge. All the stuff with axioms, I didn't see until university. This is not like the first thing you hear. So, I definitely remember the incompleteness theorems.

5. Analysis

There are many approaches one could take to analyze the statements of Julius above. However, in the current analysis, I shall try to access the "long term effects of exposure to primary historical sources" on Julius through a discussion of his development of the three types of overview and judgment (Niss & H jgaard, 2011) and relate this to Barbin's (1997) effects of being exposed to primary sources: *fonction vicariante*; *fonction dépaysante*; and *fonction culturelle*. We may visualize this be means of the following table.

Table 1: Visualization of the two theoretical constructs from Niss and Højgaard (2011) and Barbin (1997)

	Application (OJ1)	History (OJ2)	Philosophy (OJ3)
Vicariante			
Dépaysante			
Culturelle			

Of course, when combining the constructs of Niss and H jgaard (2011) and Barbin (1997) into a three by three matrix, we may not expect to be able to plot elements into every cell of the matrix. Nor should we take the number of cells "filled in" as a criterion for success of the long term effect in regard to Julius. Rather we should view it as an organized way of thinking about the kind of effects this particular HAPh-module – with its guided reading approach (Barnett et al., 2014) to the use of primary historical sources – had on Julius and his image of mathematics as a (scientific) discipline (Jankvist, 2015). I shall do the analysis column by column, i.e. one type of overview and judgment at the time.

In relation to the first type of overview and judgment (OJ1), i.e. actual applications of mathematics in other subject and practice areas, Julius' talk of "boxes" may be seen as illustrating this. Or more precisely, it illustrates an important issue regarding mathematical modeling, which is how actual applications of mathematics often come into play. Namely, that although mathematics is usually taught as belonging to specific "boxes", in real life applications, you use what makes sense and what provides you with the needed answer when building a model. Julius relays this by stating that Euler did not operate in a "predefined box". This may be seen to relate to the issue of *fonction vicariante*, since Julius clearly has realized that mathematics is not only a corpus of knowledge and techniques, but also a hotchpotch of ways for bringing these into play in given situations and contexts.

Regarding the second type of overview and judgment (OJ2), i.e. the historical evolution of mathematics, both internally and from a social point of view, Julius' talk about external and internal influences on the development of mathematics illustrates this quite well. His "evidence" in this respect is the example of minimum spanning trees, telegraph cables and computer cobber wiring. He is able to relate this example to the technological and societal context (*fonction culturelle*), when he says that: "It has to do with how time changes"; "At one point in time you need telegraph cables, later you develop the computer..."; "It has to do with the external world, how it develops, and how mathematics is brought into play." In the interviews, Julius often talks about how mathematics has developed, and that textbooks do not provide you with a correct image of this, e.g. how long time it has taken to end up at the efficient notation which is so easily used by your lecturer today. Julius says that he had not thought about this before, not until he was exposed to primary historical sources. This may be seen as an element of *fonction vicariante*. Also an element of *dépaysement* is present, since the fact that the same "mathematics can be used for different things, which are not related" appears to have made quite an impression on Julius.

Concerning the third type of overview and judgment (OJ3), i.e. the nature of mathematics as a subject (and discipline), there are clearly also issues which have left an impression with Julius. Several times he mentions the generalizing nature of mathematics, with reference to Euler's treatment of the Königsberg bridge problem, and that in mathematics one wants to reduce something to the simplest problem, and provide as general an answer as possible. Clearly this is yet an element of *fonction vicariante*. As for *dépaysement* in relation to the nature of mathematics, the story of Gödel's incompleteness theorems certainly provides estrangement for Julius. Of course, in relation to these theorems there were no primary historical sources, so ascribing this particular element of *dépaysement* to exposure of original sources is an exaggeration. Still, in combination with the Hilbert text, it clearly led to a reorientation of Julius' image of mathematics.

6. Concluding discussion

Now, when performing an analysis as that in the previous section, there are always choices to be made. For example, Julius' example of minimum spanning trees in relation to telegraph cables and computer wiring could possibly also have been ascribed to the first type of overview and judgment instead of the second. Also, further examples to fill in the cells of table 1 could possibly be found in the data (taking into account that what is displayed is only a fraction of the full interviews). But this is not the important issue here. Rather the important issue is that there *are* examples of development of all types of overview and judgment, and there *are* examples of the exposure to original sources having resulted in all three potential effects. That is to say, even three years after the completion of the course in which Julius did a project involving a reading of historical primary sources, there are "measurable" effects in regard to 1) each of the three types of overview and judgment, and 2) in regard to the issues of *fonction vicariante*, *dépaysante*, and *culturelle*. All in all, the HAPh-module appears to have had an impact on Julius' image of "mathematics as a discipline" (Jankvist, 2015), in particular by providing evidence on which to develop this image, not least in relation to the various meta-issues surrounding the case of early graph theory and its later application the shortest path problem. But what also appears evident, is that these meta-issues to some extent are *anchored* in the in-issues of the mathematical cases (Jankvist, 2011) – and my hypothesis in relation to this is that the documented long term effect is not completely unrelated to the presence of anchoring in one way or another.

Furthermore, Julius is also able to provide hindsight reflections on the design of the HAPh-module, in particular the guided reading approach – and he is able to contrast the approach of reading primary historical sources to the usual textbook teaching. One question which is often raised in relation to the use of primary sources in the teaching and learning of mathematics is whether this makes the students better mathematicians. Julius found that he did learn mathematics from the HAPh-module, but if it made him a *better* mathematician is not to say. However, what seems clear is that it made him a much more reflected student of mathematics. A student who is not only aware of the inner issues of the subject he is studying, but also the meta-perspective issues surrounding the subject, including its applications within other subject areas and disciplines, its historical evolution and development, and its science philosophical status. And my guess is that Julius is going to become an equally reflected teacher, who will be able to relay his own insights regarding the discipline of mathematics to his future students.

Σημειώσεις

1. The teaching materials may be found as texts 486 and 487 in the series 'Texts from IMFUFA': http://milne.ruc.dk/ImfufaTekster/
2. Chapter 9 in the ICMI Study on *History in Mathematics Education*; the chapter is written by Jahnke, Arcavi, Barbin, Bekken, Furinghetti, El Idrissi, da Silva and Weeks.
3. See: http://math.sfsu.edu/smith/Documents/HilbertRadio/HilbertRadio.pdf (Retrieved March 13, 2015).

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