HISTORY IN THE MATHEMATICS LABORATORY: AN EXPLORATORY STUDY

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Abstract

According to some researchers one of the ways of using history in mathematics teaching is the hermeneutic approach. In this paper we report on a classroom experiment where the teacher (A. D.) tried to apply this procedure in teaching the topic "exponential and logarithms" to 18-19 year-old students attending the final year of an Italian high school. The used historical sources are excerpts from Euler's *Introductio in analysin infinitorum.* The involved students have a low motivation to study mathematics and their learning difficulties emerged also in our experiment. This fact stressed some elements of problematization of the hermeneutic approach that will be discussed in this paper.

Key words

Mathematics teaching; historical sources; Euler; hermeneutic approach; classroom experiment; low motivation.

Περίληψη

Σύμφωνα με υπάρχουσες έρευνες ένας τρόπος αξιοποίησης της Ιστορίας στη διδασκαλία των Μαθηματικών είναι η «ερμηνευτική προσέγγιση». Σ' αυτό το άρθρο αναλύουμε μια πειραματική διδασκαλία όπου ο καθηγητής (ο πρώτος εκ των συγγραφέων) προσπαθεί να εφαρμόσει αυτή την προσέγγιση στη διδασκαλία του θέματος "εκθετικές συναρτήσεις και λογάριθμοι" με μαθητές της τελευταίας τάξης του Ιταλικού Λυκείου (18-19 ετών). Οι ιστορικές πηγές που χρησιμοποιήθηκαν είναι αποσπάσματα από το βιβλίο του Euler *Introductio in analysin infinitorum.* Οι εμπλεκόμενοι μαθητές είχαν περιορισμένα κίνητρα για τη σπουδή των Μαθηματικών και οι δυσκολίες τους στα Μαθηματικά εμφανίζονται και στην πειραματική μας διδασκαλία. Το γεγονός αυτό επισημαίνει ορισμένα στοιχεία προβληματισμού όσον αφορά στην «ερμηνευτική προσέγγιση», τα οποία συζητούνται σ' αυτό το άρθρο.

Λέξεις Κλειδιά

Διδασκαλία μαθηματικών, Ιστορικές πηγές, Euler, ερμηνευτική προσέγγιση, πειραματική διδασκαλία, περιορισμένα κίνητρα.

0. Introduction

In this paper we discuss an experiment on the use of primary sources in mathematics teaching with particular reference to the educational problem of getting students to understand, that is to say to build meanings in the mathematics classroom. In this concern Gardner (1991) is quite pessimistic. He maintains that "Even though educational systems may pay lip-service to goals like 'understanding' or 'deep knowledge' they, in fact, prove inimical to the pursuit of these goals." For this reason he has "sought to challenge the conception that one can get student to understand simply by presenting them with good models or with compelling demonstrations, as well as the idea that students who do not understand must simply work harder or adhere to the correct-answer compromise." For Gardner the students have to learn to make new, different, and strategic uses of the sources of information around them. He also claims "If we are to achieve a milieu in which understanding is prized, it is necessary for us all to be humble about what we know and to move away from our present, invariably inadequate perspectives ...".

We share Howard Gardner's concern about the understanding and we feel that our positive orientation about the use of history in the mathematics classroom may offer the new perspective we look for and new sources of information for students. As we will see, suitable conditions have to be created to achieve our goal.

Demattè (2004; 2006) has outlined two main roles of the use of history in mathematics teaching: the "strong" role and the "weak" role. The strong role is based on didactical activities that are directly inherent to history, the weak role confines the use of history to mathematics. A similar distinction is introduced by Jankvist (2009). This author considers the use of *history as a goal* and the use of *history as a tool*. The first does not serve the primary purpose of being an aid, but rather that of being an aim in some sense. In contrast, a use of history as a tool concerns the use of history as an aid in the teaching and learning of mathematics. In the same vein Furinghetti (1997) identifies two main aims of the integration of the history in the teaching of mathematics: promoting mathematics, that is acting on the image of mathematics by setting it in a wider cultural context, and reflecting on mathematics, that is dealing with mathematical concepts. This second aim refers to the fact that history brings back the modern concepts and theories presented in a polished form to their cognitive roots. With the first aim the discipline is set in the context of human civilization and broaden the scope of mathematics teaching to other disciplines. This aim supports cultural understanding. As Jahnke et al. (2000, p. 292) put it

> Integrating history of mathematics invites us to place the development of mathematics in the scientific and technological context of a particular time and in the history of ideas and societies, and also to consider the history

> of teaching mathematics from perspectives that lie outside the established disciplinary subject boundaries.

Of course, the two aims may have commonalities. In this paper we are mainly concerned with the strong role of history and the general cultural value fostered by it, nevertheless we will see that the construction of meaning for mathematical concepts will be considered too.

1. Theoretical frame

In Italy the school situation and the recommendations of the new programs encourage promoting the cultural understanding in the teaching of mathematics. Moreover these programs suggest organizing activities in the mathematics laboratory. The concept of mathematics laboratory is an old one. Already in 1895 Adelia Roberts Hornbrook wrote a pamphlet (*Laboratory methods of teaching mathematics in secondary schools.* New York: Cincinnati American Book Company) focused on this subject. The idea developed in the first decades of twentieth century elsewhere, especially in France and UK, see (Giacardi, 2013). In subsequent years it was revitalized in connection with the interest for manipulatives and concrete materials, see (Furinghetti & Menghini, 2014). When computers appeared in mathematics education the mathematics laboratory acquired a new role and importance, see (Hynes, Hynes, Kysilka & Brumbaugh, 1973). Parallelly to the view of laboratory as a place where to carry out bodily activities with manipulatives and, later on, with information technology, the view emerged that the mathematics laboratory

> is not a physical space outside the classroom, but is rather a structured set of activities aimed at constructing the *meanings* of mathematical objects. Thus, the laboratory involves people (students and teachers), structures (classrooms, instruments, organization of spaces and times), and ideas (projects, plans for educational activities, experimentation). (Matematica 2003, p. 28, our translation).

The mathematics laboratory philosophy is in line with the actual trend in mathematics education that advocates the active participation of students and the interaction in the classrooms, so that the students become the main characters of the mathematics classroom.

The experiment reported in this paper shows how history of mathematics can be used with the aim of fulfilling the indications of the new Italian programs reported above, and, in particular, with the aim of making the mathematics laboratory a place where to develop the cultural understanding.

Among the researchers investigating on the use of history there is a solid agreement on the pedagogical efficacy of the use of original historical sources in mathematics teaching, see Furinghetti, Jahnke & van Maanen, 2006; Jankvist, 2014; Pengelley, 2011). For our purposes in the classroom we considered some experiment that was based on the hermeneutic approach to original sources, see (Bagni, 2008; Glaubitz, 2011).

Jahnke (2014, pp. 83-84) outlines the basic guidelines of the hermeneutic procedure as follows:

1. Students study a historical source *after* they have acquired a good understanding of the respective mathematical topic in a *modern* form and a *modern perspective*. The source is studied in a phase of teaching when the new subject-matter is applied and technical competencies are trained. Reading a source in this context is another manner of applying new concepts, quite different from usual exercises.
2. Students gather and study information about *context* and *biography* of the author.
3. The historical *peculiarity* of the source is kept as far as possible.
4. Students are encouraged to produce *free associations*.
5. The teacher insists on *reasoned arguments*, but not on accepting an interpretation which has to be shared by everybody.
6. The historical understanding of a concept is contrasted with the modern view, that is the source should encourage processes of reflection".

These points make clear the potentialities of the method. The students have the chance of getting in touch with the historical method (points 2 and 3), of making conjectures and proving them (points 4 and 5), and of reflecting on modern notations by making sense of historical mathematical text (points 1 and 6).

2. Method

The experiment was carried out in the final year of a "Liceo delle scienze umane" (Lyceum of human sciences), with 20 students (three boys, 17 girls) aged 18-19. The students worked in pair formed spontaneously by the students themselves and at the end presented an individual report. The ideas put forwards were discussed during the sessions. The activities were carried out in the classroom, where there was one computer and Internet facilities. The school has a humanistic orientation and the students have poor motivation towards mathematics. Since the first year of upper secondary school their curriculum included the innovations contemplated by the Italian reform launched in 2010-2011. This reform is not changing so much the existing situation, nevertheless it offers some chances to carry out some non-traditional activities. For example, in the third, fourth, and fifth year there are two hours per week devoted to deepening a subject chosen by students, and a subject (not linguistic, such as Latin or foreign languages) is taught in a language of the European community by the teacher with a mother language assistant (at least 20 hours per year). In our experiment the subject taught in the foreign language (English) was mathematics.

In the activity here reported the teacher is one of the authors (A. D.) In the previous years the students had a different teacher, then the methods and the style of the teaching (including his orientation towards a historical perspective) of the present teacher were new for them.

The mathematical topic developed was the function concept. He has planned to present other topics of Calculus in the last part of the course, compatibly with the reduction of the amount of hours dedicated to mathematics. The students were presented with historical documents dealing with topics suitable to prepare to the topic "function". The stages of the work were the following:

- Presentation of the task, which consisted in interpreting a historical source (Euler, Appendix 1)
- Two hours dedicated to working on the task
- Suggestions for making students going on, when the students did not achieve significant results (which was our case)
- A first test containing questions after reading the documents (Appendix 3)
- Further questions to prepare students to the assessment test (Appendix 4)
- Final test for assessing students (Appendix 5).

The proposed original source was some excerpts from *Introductio in analysin infinitorum* (Euler, 1748) translated into English by J. D. Blanton (1988), see Appendix 1. The fact that the students were attending these lessons in English turned to be an element that favored the development of the experiment. The teacher stressed that the main goal of the activity was exploiting the mathematics they have learned in a new way and using personal resources to conjecture and explore.

The given task was to read and interpret the historical passages, e.g. to write the document in the current Italian, to guess about the meaning of the full documents and its specific parts and to compare the individual interpretations through the discussion. The teacher assisted the work, listened to students' questions, and answered through hints or other questions that could help them to reflect. In order to give further opportunities for better understanding and for reviewing all contents, he suggested comparing their answers: standing in front of their classmates, most of them read at least one of their answers; they could also briefly criticize the mates' outputs.

As stated in the Preface reported in Appendix 2 Euler's treatise was originally conceived for beginners. Unfortunately in our experiment most students did not have the necessary prerequisites; this was the case of some of them (Anna, Alessandro, Chiara and Valentina). For example, in treating exponential and logarithmic Euler introduces in an informal manner examples of limits. This fact may have contaminated the hermeneutic approach (see guideline 1 of Jahnke (2014)), since the students still had some difficulties about these concepts.

The request of interpreting the document was addressed to all students, but, according to the students' different needs, with different aims: remedial for lacking notions, deepening of concepts, application of concepts which are different from the usual exercises, ...).

Students were asked to perform further tasks such as gathering information about the context and the biography of the historical author. This was made through the website, but in a rather inaccurate way.

3. Outcomes and discussion

Lampert (1990) claims that "When a student is in charge of revising his or her own thinking, and expected to do so publicly, the authority for determining what is valid knowledge is shifted from the teacher to the student and the community in which the revision is asserted". (p. 52) Then, in principle, the hermeneutic approach carried out in the way described above should foster the devolution of authority from the teacher to the students. In practice, the students did not show significant disposition to produce free associations. Our interpretation of this behavior refers to the students' low self-confidence linked to their view of classroom life. In their school biography there are not teaching situations in which putting forwards personal ideas and producing free associations and personal conjectures may turn to be an occasion for getting good marks. The consequence is that the students were asking themselves: which rules their performances should follow; whether their rules were matching the teacher's rules? When they met difficulties to understand the text, either because of the English language, and/or their poor practice to read mathematical texts, they could not decide about the validity of their performance. For example, a doubt was whether their understanding of the text was inadequate because they were not able to answer the questions generated by themselves about specific parts of the document.

In this context, the historical source was used just as a script for realizing performances operational for what they were thinking could be the teacher's expectations. The teacher and researcher's aims were: to recover and to deepen the meaning of concepts through the use of the historical source; to enhance the ability in argumentation by asking themselves questions originated from the historical document trying to answer through reflection and discussion with the other groups. These aims were not considered by the students for their perception of the didactic contract.

In summary we may say that the task we gave to the students may be understood in two ways:

- To talk about the document so that the document becomes an object of study. For example, the student may note that the document presents an approach to the concept of exponential and logarithm and may focus on particular aspects of this issues
- To rephrase the historical document by using modern symbols and approach.

The comparison of the excerpts from Euler's treatise (Appendix 2) with the text produced by a student in answering to the questions proposed by the teacher (Appendixes 5, 5′, 5′′) shows that the second way was taken into consideration. As discussed by Arcavi and Isoda (2007) the translation to modern notation was a useful strategy for making sense of the historical document. The need of working in the classroom with the aim of passing examinations fostered an interpretation of the task as a temporary action towards the stage in which the interpretation was "shared by everybody".

The task given to the students was accompanied by the requirement: "When performing the task, record on the sheet also your abandoned ideas, failed attempts etc. This record will be used in our classroom discussion. If you want to erase what you have written, please just strike through so that the original text may be seen". This means that the students were asked to describe as much as possible their process (including unsuccessful attempts) of approaching the historical document. It was expected that the groups would provide different outcomes. Unfortunately the students did not care of the suggestion in the task. In front of the new situation presented by the hermeneutic approach they preferred to negotiate some aspects of the didactic contract. Their negotiation was based on the questions about the rules to be followed and the teacher's expectation. It was carried out by some students selected as "spokespersons" and through a classroom discussion. The following sentence by a student expresses the general feeling:

This is an easy document. I do not know what to say about it and how to rephrase it.

4. Reflections and preliminary conclusions

Our experiment has an exploratory character and allows us to outline some open questions that may be considered in future research and in our future practice in the classroom. The first question concerns the cognitive value of the concepts encountered in Euler's document. In other words are the students ready to use these concepts in other contexts? How they feel the sense of commonality with other persons and with the modalities these persons (mates or adults) learn the same topics? The main underlying question is: do the students consider Euler an added value to their mathematics learning or simply an oddity introduced by the teacher.

About the research on the use of history in mathematics teaching some members of the HPM Study Group advocate a strict link between pedagogical and historical issues, see (Jankvist, Mosvold, Fauskanger & Jakobsen, in press). We deem that our experiment goes in this direction, since our analysis and discussion show that the outputs of our experiment are strongly dependent on the conditions affecting it. In our case the beliefs about mathematics learning, didactic contract, and devolution of authority have affected the behavior of students facing the task involving the historical document. These issues are fundamental issues in research in mathematics education. The analysis reported in this study shows how the discussion on the use of history acquires meaning and efficiency if it is set in the suitable framework of mathematics education research.

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http://www.21learn.org/archive/review-the-unschooled-mind-by-howard-gardner/

Appendix 1. Excerpts from Introductio in analysin infinitorum (Euler, 1748)

97. Let the exponential to be considered be a^z where a is a constant and the exponent z is a variable. Since the exponent z stands for all determined numbers, it is clear at least that all positive integers can be substituted for z to give determined values a^1, a^2, a^3, a^4, a^5, a^6, etc. If for z we substitute the negative integers –1, –2, –3, etc., we obtain $\frac{1}{a}$, $\frac{1}{a^2}$, $\frac{1}{a^3}$, $\frac{1}{a^4}$, etc. If $z = 0$, then we have $a^0 = 1$. If we substitute a fraction for z, for instance $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}$, etc. we obtain the values $\sqrt{a}$, $a^{\frac{1}{3}}$, $a^{\frac{2}{3}}$, $a^{\frac{1}{4}}$, $a^{\frac{3}{4}}$, etc.

98. The values of the exponential a^z depend primarily on the magnitude of the constant a. If $a = 1$, then we always have $a^z = 1$, no matter what value is given to z. If $a > 1$, then a^z will have a greater value if the value of z is greater than it was originally and as z goes to infinity, so also a^2 increases to infinity. If $z = 0$, then $a^z = 1$; if $z < 0$, then the values of a^z become less than 1 and as z goes to $-\infty$, a^z goes to 0. On the other hand if $a < 1$ but still positive, then the values of a^z decrease when x increases above 0. The exponential increases as z increases in the negative direction. Since when $a < 1$, we have $\frac{1}{a} > 1$, and if we let $\frac{1}{a} = b$, then $a^z = b^{-z}$. For this reason we can examine the case when $a < 1$ from the case when $a > 1$.

99. If $a = 0$, we take a huge jump in the values of a^z. As long as the value of z remains positive, or greater than zero, then we always have $a^z = 0$. If $z = 0$, then $a^0 = 1$. However if z is a negative number, then a^z takes on an infinitely large value; for example, if $z = -3$, then $a^z = 0^{-3} = \frac{1}{0^3} = \frac{1}{0}$,

which is infinite. Much greater jumps occur if the constant a takes on a negative value, for instance – 2. In this case, when z takes on integral values, a^z takes positive and negative values alternately, as can be seen from the sequence a^{-4}, a^{-3}, a^{-2}, a^{1}, a^{0}, a^{1}, a^{2}, a^{3}, a^{4}, etc.

$+\frac{1}{16}$, $-\frac{1}{8}$, $+\frac{1}{4}$, $-\frac{1}{2}$, + 1, –2, + 4, –8, + 16. Furthermore if the exponent z takes fractional values, then $a^z = (-2)^z$ sometimes has real values and sometimes complex values. For instance a $= \sqrt{-2}$ which is a pure imaginary, while $a^{\frac{1}{3}} = (-2)^{\frac{1}{3}} = -2^{\frac{1}{3}}$ which is real. If the exponent z is given an irrational value, then a^z may give real or complex values, but this cannot be predicted.

100. After having considered the inconveniences associated with a negative value

for *a*, we decide that *a* will be a positive number, indeed greater than 1, since from this case it is easy to investigate the case when *a* lies between 0 and 1. If we let $y = a^z$, and for *z* substitute all real numbers, which lie between $-\infty$ and $+\infty$, then *y* takes all positive real values between 0 and $+\infty$. If *z* goes to ∞, then *y* also goes to ∞, if $z = 0$, then $y = 1$ and when *z* goes to $-\infty$, *y* goes to 0. On the other hand, for any positive value assigned to *y*, there is a real value corresponding to *z* such that $a^z = y$. If a negative value is given to *y*, there is no corresponding real value for *z*.

101. If $y = a^z$, then *y* is a function of *z*, and the extent to which *y* depends on *z* is easily understood from the nature of exponents. Thus whatever value is given to *z*, the value of *y* is determined. For instance

$y^2 = a^{2z}$, $y^3 = a^{3z}$ and generally $y^n = a^{nz}$. From this it follows that $\sqrt{y} = a^{\frac{1}{2}z}$, $a^{\frac{1}{3}} = a^{\frac{1}{3}z}$, and $\frac{1}{y} = a^{-z}$, $\frac{1}{y^2} = a^{-2z}$, $\frac{1}{\sqrt{y}} = a^{-\frac{z}{2}}$, and so forth.

Furthermore, if $v = a^x$, then $vy = a^{x+z}$ and $\frac{v}{y} = a^{x-z}$. A benefit we derive *y* from these properties is that it is easier to determine the value of *z* when a value of *y* is given.

EXAMPLE

If $a = 10$, from arithmetic, which we shall use, the number ten makes it easy to see the values of *y* when we substitute values for *z*. We see that $10^1 = 10$, $10^2 = 100$, $10^3 = 1000$, $10^4 = 10000$, and $10^0 = 1$. Likewise $10^{-1} = \frac{1}{10} = 0.1$, $10^{-2} = \frac{1}{100} = 0.01$, $10^{-3} = \frac{1}{1000} = 0.001$. If we let *z* have fractional values, by means of root extraction, we can state the values of *y*. Thus $10^{\frac{1}{2}} = \sqrt{10} = 3.162277$, etc.

102. Just as, given a number *a*, for any value of *z*, we can find the value of *y*, so, in turn, given a positive value for *y*, we would like to give a value for *z*, such that $a^2 = y$. This value of *z*, insofar as it is viewed as a function of *y*, it is called the LOGARITHM of *y*. The discussion about logarithms supposes that there is some fixed constant to be substituted for *a*, and this number is the base for the logarithm. Having assumed this base, we say the logarithm of *y* is the exponent in the power a^z such that $a^z = y$. It has been customary to designate the logarithm of *y* by the symbol $\log y$. If $a^z = y$, then $z = \log y$. From this we understand that the base of the logarithms, although it depends on our choice, still it should be a number greater than 1. Furthermore, it is only of positive numbers that we can represent the logarithm with a real number.

103. Whatever logarithmic base we choose, we always have log 1 = 0, since in the equation $a^z = y$, which corresponds to $z = \log y$, when we let $y = 1$ we have $z = 0$. From this it follows that the logarithm of a number greater than 1 will be positive, depending on the base a. Thus $\log a = 1$, $\log a^2 = 2$, $\log a^3 = 3$, $\log a^4 = 4$, etc. and, after the fact, we know what base has been chosen, that is the number whose logarithm is equal to 1 is the logarithmic base. The logarithm of a positive number less than 1 will be negative. Notice that $\log \frac{1}{a} = -1$, $\log \frac{1}{a^2} = -2$, $\log \frac{1}{a^3} = -3$, etc., but the logarithms of negative numbers will not be real, but complex, as we have already noted.

104. In like manner, if $\log y = z$, then $\log y^2 = 2z$, $\log y^3 = 3z$, etc. and in general $\log y^n = nz$ or $\log y^n = n \log y$, since $z = \log y$. It follows that the logarithm of any power of y is equal to the product of the exponent and the logarithm of y. For example, $\log \sqrt{y} = \frac{1}{2} z = \frac{1}{2} \log y$,

$\log \frac{1}{\sqrt{y}} = \log y^{-\frac{1}{2}} = -\frac{1}{2} \log y$, and so forth. It follows that if we know the logarithms of any number, we can find the logarithms of any power of that number. If we already know the logarithms of two number, for example $\log y = z$ and $\log v = x$, since $y = a^z$ and $v = a^x$, it follows that $\log vy = x + y = \log v + \log y$. Hence, the logarithm of the product of two numbers is equal to the sum of the logarithms of the factors. In like manner, $\log \frac{y}{v} = z - x = \log y - \log v$, that is, the logarithms of a quotient is equal to the logarithm of the numerator diminished by the logarithm of the denominator. These rules can be used to find the logarithms of many numbers from a knowledge of the logarithms of a few.

Leonhard Euler, 1988, *Introduction to Analysis of the Infinite,* New York: Springer; book 1, pp. 75-80.
Translation of: *Intruductio in analysin infinitorum,* 1748.

[In the last paragraph of the original we find “$\log vy = x + \underline{y} = \log v + \log y$”. The underlined letter has to be read “z”]

Appendix 2. Preface of *Introductio in analysin infinitorum* (Euler, 1748)

PREFACE

Often I have considered the fact that most of the difficulties which block the progress of students trying to learn analysis stem from this: that although they understand little of ordinary algebra, still they attempt this more subtle art. From this it follows not only that they remain on the fringes, but in addition they entertain strange ideas about the concept of the infinite, which they must try to use. Although analysis does not require an exhaustive knowledge of algebra, even of all the algebraic techniques so far discovered, still there are topics whose consideration prepares a student for a deeper understanding. However, in the ordinary treatise on the elements of algebra, these topics are either completely omitted or are treated carelessly. For this reason, I am certain that the material I have gathered in this book is quite sufficient to remedy that defect. I have striven to develop more adequately and clearly than is the usual case those things which are absolutely required for analysis. Moreover, I have also unraveled quite a few knotty problems so that the reader gradually and almost imperceptibly becomes acquainted with the idea of the infinite.

Leonhard Euler, 1988, *Introduction to Analysis of the Infinite*, New York: Springer; book 1, Preface.
Translation of: *Intruductio in analysin infinitorum*, 1748.

Appendix 3. Intermediate test

100.

a) Solve these equations: $10^x = 0.1$ $5^x = 2$ $3^x = -1$

b) Let $a > 1$, then complete the missing parts in the following cases:

$$\lim_{x \to +\infty} a^x = \cdots$$

$$\lim_{x \to -\infty} a^x = \cdots$$

c) In class we have analyzed some properties of a^x when $0 < a < 1$. Euler indeed explains the case a > 1! Analyze it using examples, graphs, equations etc.

101.

a) "… $y^n = a^{nz}$. From this it follows that $\sqrt{y} = a^{\frac{1}{2}z}$ … $\frac{1}{\sqrt{y}} = a^{-\frac{z}{2}}$ …". How do you explain this sentence?

102.

a) How does Euler define *logarithm*? If $a^z = y$, then log y = … (complete the missing part).

b) What does he say about the base of logarithms? Let a the base of logarithm: does Euler consider the case $0 < a < 1$?

103.

a) Explain why log 1 = 0.

b) Positive/negative values of logarithm; logarithm of positive/negative numbers: how does Euler describe this various situations?

104.

Complete the following sentences using words and symbols, as well:

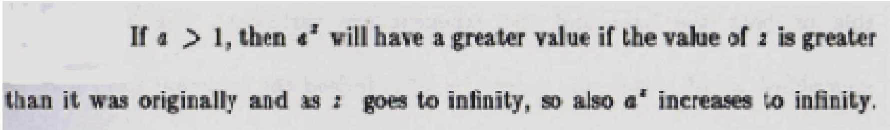
a) The logarithm of a power is equal …

b) The logarithm of a product of two numbers is equal …

c) The logarithm of a quotient is equal …

Appendix 4. Final test

a) *[2 points]* Briefly explain the following sentence using numeric examples, words, a graph and symbols.



If $a > 1$, then a^z will have a greater value if the value of z is greater than it was originally and as z goes to infinity, so also a^z increases to infinity.

b) *[1 point]* Let $a < 1$, then complete the missing parts in the following cases:

$$\lim_{x \to +\infty} a^x = \cdots$$

$$\lim_{x \to -\infty} a^x = \cdots$$

Explain your choices.

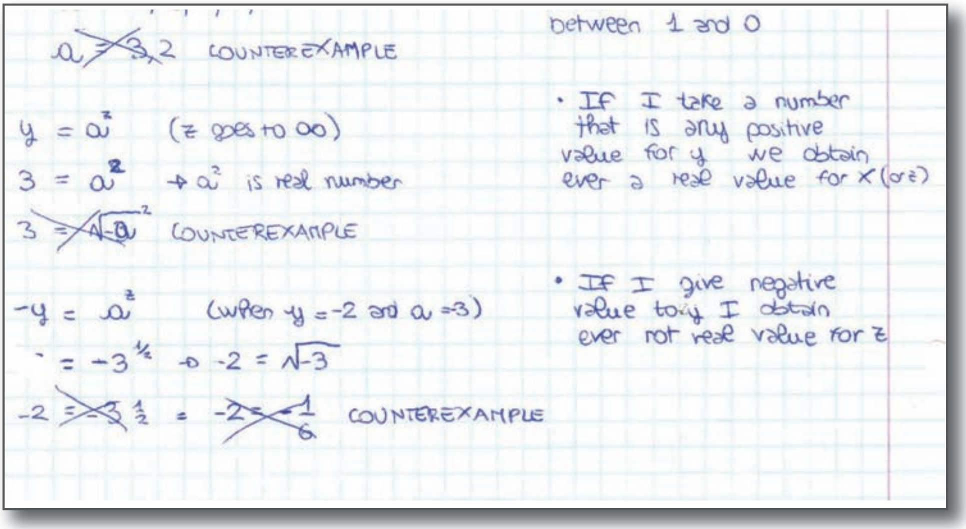
c) *[1 point]* Write the Euler's definition of *logarithm* using: *y*, the exponent *z* and the base *a*.

d) *[1 point]* Explain the following properties of logarithms using symbols and numeric examples: I. logarithm of a power; II. logarithm of a product; III. logarithm of a quotient.

e) *[5 points]* [In Italian in the original. This question doesn't regard Euler's document] Domain, intersection with x/y axis, coordinates of at least three more points, sign and presumed graph of the function:

$$f(x) = \frac{(x-3)(x+2)}{x^2}$$

Appendix 5. A student's protocols (question 100)



$a \cancel{= 3}, 2$ COUNTEREXAMPLE

$y = a^z$ (z goes to ∞)

$3 = a^2 \to a^2$ is real number

$3 \cancel{= \sqrt{-a}}^{2}$ COUNTEREXAMPLE

$-y = a^z$ (when $y = -2$ and $a = -3$)

$= -3^{1/2} \to -2 = \sqrt{-3}$

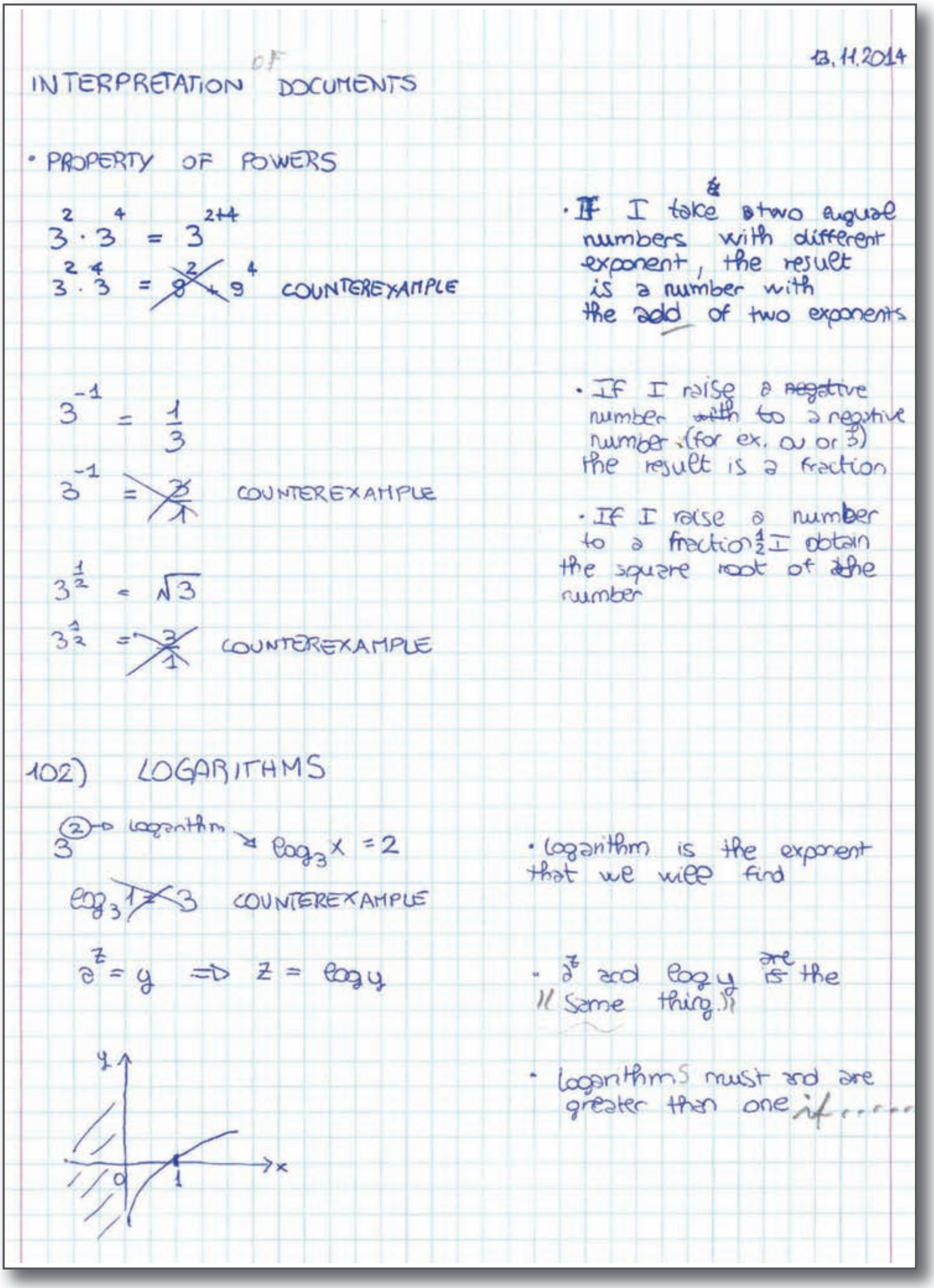
$-2 \cancel{= -3}\, \frac{1}{2} = \cancel{-2 = -\frac{1}{6}}$ COUNTEREXAMPLE

between 1 and 0

- If I take a number that is any positive value for y we obtain ever a real value for x (or z)

- If I give negative value to y I obtain ever not real value for z

Appendix 5′. A student's protocols (questions 101, 102)



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INTERPRETATION OF DOCUMENTS

- PROPERTY OF POWERS

$3^{2} \cdot 3^{4} = 3^{2+4}$

$3^{2} \cdot 3^{4} = \cancel{8^{2} +} 9^{4}$ COUNTEREXAMPLE

$3^{-1} = \frac{1}{3}$

$3^{-1} = \cancel{\frac{3}{1}}$ COUNTEREXAMPLE

$3^{\frac{1}{2}} = \sqrt{3}$

$3^{\frac{1}{2}} = \cancel{\frac{3}{1}}$ COUNTEREXAMPLE

- If I take & ~~a~~ two augual numbers with different exponent, the result is a number with the add of two exponents

- If I raise a ~~negative~~ number ~~with~~ to a negative number (for ex. ω or 3) the result is a fraction

- If I raise a number to a fraction $\frac{1}{2}$ I obtain the square root of the number

102) LOGARITHMS

$3^{\textcircled{2}} \to$ logarithm $\searrow \log_3 x = 2$

$\log_3 \cancel{1 \neq} 3$ COUNTEREXAMPLE

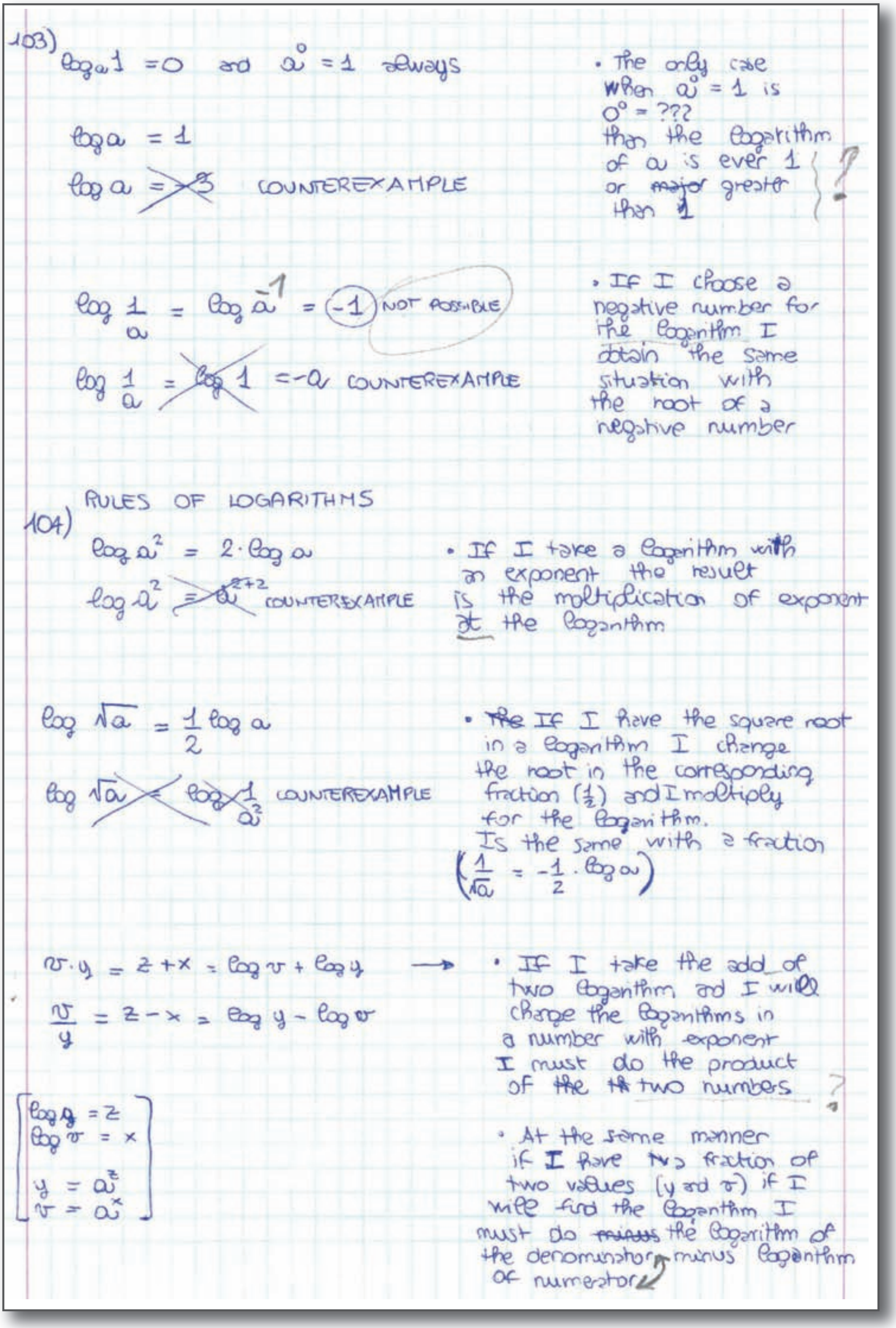
$a^{z} = y \Rightarrow z = \log y$

- logarithm is the exponent that we will find

- a^{z} and $\log y$ ~~is~~ are the // same thing.!!

- logarithms must and are greater than one if......

Appendix 5″. A student's protocols (questions 103, 104)



103) $\log_a 1 = 0$ and $a^0 = 1$ always

$\log a = 1$

$\log a \cancel{= 3}$ COUNTEREXAMPLE

- The only case when $a^0 = 1$ is $0^0 = ???$ than the logarithm of a is ever 1 or ~~major~~ greater than 1 } ?!

$\log \frac{1}{a} = \log a^{-1} = \boxed{-1}$ NOT POSSIBLE

$\log \frac{1}{a} = \cancel{\log 1} = -a$ COUNTEREXAMPLE

- If I choose a negative number for the logarithm I obtain the same situation with the root of a negative number

RULES OF LOGARITHMS

104) $\log a^2 = 2 \cdot \log a$

$\log a^2 \cancel{= a^{2+2}}$ COUNTEREXAMPLE

- If I take a logarithm with an exponent the result is the moltiplication of exponent at the logarithm

$\log \sqrt{a} = \frac{1}{2} \log a$

$\log \sqrt{a} \cancel{= \log \frac{1}{a^2}}$ COUNTEREXAMPLE

- ~~The~~ If I have the square root in a logarithm I change the root in the corresponding fraction ($\frac{1}{2}$) and I moltiply for the logarithm. Is the same with a fraction $\left(\frac{1}{\sqrt{a}} = -\frac{1}{2} \cdot \log a\right)$

$v \cdot y = z + x = \log v + \log y$ ⟶

$\frac{v}{y} = z - x = \log y - \log v$

$$\begin{bmatrix} \log y = z \\ \log v = x \\ \\ y = a^z \\ v = a^x \end{bmatrix}$$

- If I take the add of two logarithm and I will change the logarithms in a number with exponent I must do the product of the ~~th~~ two numbers ?
- At the same manner if I have two fraction of two values (y and v) if I will find the logarithm I must do ~~minus~~ the logarithm of the denominator minus logarithm of numerator