TEACHING *WITH* AND *ABOUT* THE NATURE OF MATHEMATICS *THROUGH* THE HISTORY OF MATHEMATICS: ENACTING INQUIRY LEARNING *IN* MATHEMATICS

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Abstract

The purpose of the present paper is two-fold: 1) to explore how university (teacher) students of mathematics, through history (and philosophy) of mathematics, can develop informed conception about the epistemology of mathematics, of how mathematicians produce mathematical knowledge, what kind of questions that drive mathematical research and how mathematical research is validated, while experiencing an inquiry learning environment that gives them insight into authentic mathematical practice in mathematics as a science; and 2) to illustrate how such students are capable of teaching in upper secondary school with and about the nature of mathematics within an explicit-reflective framework in the sense of the American educator Fouad Abd-El-Khalick (2013) by endorsing learning environments that to some extent and in some sense bear resemblance to authentic mathematical practices, that is, by establishing inquiry learning environments in mathematics as a science.

Key words

History of mathematics, epistemology of mathematics, inquiry based teaching and learning, problem oriented project work, student project work in mathematics.

Περίληψη

Ο σκοπός του παρόντος άρθρου είναι διπλός: 1) Να εξετάσει τη συμπεριφορά φοιτητών μαθηματικών (μελλοντικών καθηγητών) που εργάζονται σε ένα περιβάλλον διερευνητικής μάθησης, το οποίο τους παρέχει την δυνατότητα επίγνωσης των αυθεντικών πρακτικών που χρησιμοποιούνται στη μαθηματική επιστήμη. Ειδικότερα διερευνάται το πως μπορούν φοιτητές που δουλεύουν σε ένα τέτοιο περιβάλλον, μέσω της ιστορίας (και της φιλοσοφίας) των μαθηματικών, να διαμορφώσουν τεκμηριωμένη αντίληψη όσον αφορά στην επιστημολογία των μαθηματικών, στο πώς οι μαθηματικοί παράγουν μαθηματική γνώση, στο είδος των ερωτήσεων που οδηγεί την μαθηματική

έρευνα και στο πώς η μαθηματική έρευνα επικυρώνεται και αξιολογείται. 2) Να καταδείξει πώς τέτοιοι φοιτητές είναι ικανοί να διδάξουν στο Λύκειο με και περί της φύσης των μαθηματικών εντός ενός ρητά αναστοχαστικού πλαισίου, υπό την έννοια του Fouad Abd-El-Khalick (2013), ενισχύοντας μαθησιακά περιβάλλοντα τα οποία σε κάποιο βαθμό υποστηρίζουν ομοιότητες με αυθεντικές μαθηματικές πρακτικές, δηλαδή, δημιουργώντας περιβάλλοντα διερευνητικής μάθησης όσον αφορά στη μαθηματική επιστήμη.

Λέξεις κλειδιά

Ιστορία των μαθηματικών, επιστημολογία των μαθηματικών, διερευνητική διδασκαλία και μάθηση, σχέδιο εργασίας προσανατολισμένο στην επεξεργασία προβλημάτων, σχέδιο εργασίας φοιτητών στα μαθηματικά.

Introduction

The Rocard (2007) report ascribed the decline of interest in science and mathematics among young people in Europe to the way in which these subjects are taught in schools. In accordance with this, the first finding listed in the report is that:

> A reversal of school's science teaching pedagogy from mainly deductive to inquiry-based methods provides the means to increase interest in science.[1] (Rocard 2007, p. 12)

The underlying assumption is that by engaging students in scientific inquiry learning activities i.e., activities that in some sense mimic or bear resemblance to how scientists work, they will get a deeper integrated understanding of science and scientific knowledge; which in turn will foster motivation and interest.

The recommendation for inquiry based pedagogy has created a host of EU funded projects with the aims of promoting inquiry based teaching in science– and in several of these mathematics is included as a science subject (see the website www.scientix.eu). With these projects, inquiry based teaching and learning has migrated into mathematics education. In the Rocard report they use Linn, Davis, and Bell's (2004) definition of inquiry:

> inquiry is the intentional process of diagnosing problems, critiquing experiments, and distinguishing alternatives, planning investigations, researching conjectures, searching for information, constructing models, debating with peers, and forming coherent arguments. (Rocard 2007, p. 9)

Which indeed is a description of a process that bears resemblance to how scientists work when they create scientific knowledge. The connection to mathematics education is created by reference to problem solving:

> In mathematics teaching, the education community often refers to "Problem-BasedLearning" (PBL) rather than to IBSE [inquiry based science education]. (Rocard 2007, p. 9)

And, indeed, problem solving and mathematical modeling play a major role in inquiry-based mathematics education [see e.g. the survey paper by Artigue and Blomh j (2013)].

Inquiry-based teaching aims at inviting students into the workplace of scientists and mathematicians. The idea is that if students are engaged in activities and learning processes similar to the way scientists and mathematicians produce knowledge and work with science and mathematics, the students will develop a deep understanding of science and mathematics, as well as of the epistemology and more broadly the nature of these subjects. However, the emphasis on mathematical modeling and real life problem solving may provide students with a limited picture of mathematics and of what mathematicians do when they create new mathematics, how they do it, why they do it, where they got their ideas from, if these ideas are 'good' ideas, whether the produced mathematics is 'good' or 'bad', how the mathematicians distinguish 'good' from 'bad' and what characterizes fruitful mathematics. In other words: it is not clear how students, through these kinds of activities, will experience and develop informed conceptions about the epistemology of mathematics, of how mathematicians produce mathematical knowledge, what kind of questions that drive mathematical research and how mathematical research is validated.

The didactician of mathematics, Mogens Niss (1994, p. 367) distinguishes between five faces of mathematics that constitute mathematics as a discipline: Mathematics as 1) a pure science, 2) an applied science, 3) a system of instruments, 4) a field of aesthetics and 5) a teaching subject. Characteristics for the first two faces are that within these, the aim of mathematics is to produce knowledge about and insights into objects, relations and theory-buildings within mathematics. While problem solving activities can relate to mathematics as a pure science, mathematical modeling activities mainly relates to mathematics as a system of instruments for decisions and actions i.e., the third face.

In the present paper we will discuss how university (teacher)[2] students can be taught *about*the nature of mathematics through history of mathematics in an inquiry learning environment that to some extent simulate authentic research practices in mathematics as a science, i.e., within face 1 and face 2 above, where the aim is to produce knowledge in mathematics. We will also present an example from upper secondary school of how (teacher) students that have integrated knowledge of NOM (nature of mathematics) can use their NOM-understandings to teach mathematics in a learning environment that mimic mathematicians' production and validating of mathematical knowledge, i.e., are able to teach *with* (and actually also *about*) the nature of mathematics.

In other words: the purpose of the present paper is two-fold: 1) to explore how university (teacher) students of mathematics through history (and philosophy) of mathematics can develop informed conception about the epistemology of mathematics, of how mathematicians produce mathematical knowledge, what kind of questions drive mathematical research and how mathematical research is validated, while experiencing an inquiry learning environment that gives them insight into authentic mathematical practice in mathematics as a science (face 1 and 2 above); and 2) to illustrate how such students are capable of teaching in upper secondary school *with* and *about* the nature of mathematics within an explicit-reflective framework in the sense of the American educator Fouad Abd-El-Khalick (2013) by endorsing learning environments that to some extent and in some sense bear resemblance to authentic mathematical practices, that is, by establishing inquiry learning environments in mathematics as a science.

1. Teaching with and about the nature of mathematics within an explicit-reflective framework

Loosely speaking, inquiry-based teaching means to invite students to work in ways that are analogous or similar to how scientists and mathematicians work. In this paper, we are concerned with mathematics as a science, and in this ball game the aim of mathematicians' work is to generate knowledge about and insights into objects, relations and theories within mathematics – in short, to produce mathematical knowledge. In this setting, inquiry-based teaching means to establish a learning environment that function as a surrogate for authentic mathematical research practices. It goes without saying that teachers in order to be able to enact such learning environments must themselves have informed conception about the nature of mathematics and (processes of) mathematical research. Mathematics teaching in higher education traditionally follows the recipe: a professor gives a lecture where he/she presents some definitions and state and prove some theorems on the black board. Problems within mathematics are stated and worked out in order to illustrate the use and power of the theorems. The students' homework is focused on working out further mathematical theoretical problems and proofs. It is implicitly assumed that by being immersed in such problem solving activities into the subject matter of mathematics, its concepts, definitions, theorems, techniques and theories, students will somehow automatically absorb and develop informed conceptions about the nature of mathematics. However, in order for this to happen, issues related to the nature of mathematics needs to be addressed directly, they need to be made explicit objects of students' reflection. Inquiry experiences in themselves are not enough.[3] In order for students to develop understandings about the epistemology of mathematics, of how mathematicians produce mathematical knowledge, what kind of questions that drive mathematical research and how mathematical research is validated, they need to reflect explicitly on such experiences.

That students do not necessarily develop informed NOM (nature of mathematics) conception about mathematics as a knowledge generating scientific enterprise by being immersed in inquiry activities does not mean that inquiry cannot be used in that respect. But, as Abd-El-Khalick and Lederman (2000) have argued for the development of NOS (nature of science), an explicit-reflective framework it required in order to achieve such understandings. By 'explicit' they mean that some specific learning objectives related to students' understanding of NOS must be included in the curriculum; by 'reflective' they refer to instructions aimed at helping students to reflect upon their experiences with learning science from within an epistemological framework.

Abd-El-Khalickdistinguishes between teaching *with* and teaching *about* NOS:

> Teaching *about* NOS refers to instruction aimed at enabling students to achieve learning objectives focused on informed epistemological understandings about the generation and validation of scientific knowledge and the nature of the resultant knowledge. [...] Teaching *with* NOS entails designing and implementing science learning environments that take into consideration these robust epistemological understandings about the generation and validation of scientific knowledge. (Abd-El-Khalick, 2013, p. 2090).

If we adopt this framework to mathematics, teaching *about* the nature of mathematics means to teach towards learnings objectives of the following kind: To make students able to investigate and analyze the methods mathematicians use to generate knowledge, to discuss, criticize and asses the epistemological status of this knowledge, to investigate and analyze the role of proofs in mathematics, to investigate and discuss mathematical objects' ontological status, to investigate and discuss the relationship between mathematics and other sciences, to discuss and critically assess the distinct nature of mathematics, as well as its development historically and in interaction with culture, society and other sciences.

In order to couple the development of students' informed conceptions about the nature of mathematics with inquiry teaching in mathematics in a meaningful way, mathematics teachers must 1) have knowledge and informed conceptions of and *about* the nature of mathematics themselves, 2) come to understand how inquiry is in fact conducted in mathematical research, 3) be able to design inquiry based learning environments *in* mathematics teaching, and 4) be able to teach *with* the nature of mathematics in the sense explained above for nature of science.

2. Inquiry teaching in mathematics through history of mathematics

How can (teacher) students in higher mathematics educations get firsthand experiencesin a meaningful way with research practices in a field that is so specialized and operates with such abstract notions as mathematics, where students usually are not invited into a research environment until they become Ph.D. students? There is not much help to gain from their textbooks. Mathematics textbooks very rarely discuss or indicate where the mathematical objects they are dealing with come from, why they look the way they do and why they are interesting. On the contrary, mathematical objects are usually introduced as timeless entities that appear in textbooks seemingly out of nowhere and for no particular reason. Most of the mathematics that students are introduced to at bachelor level and in the first year of master programs has not been developed recently. For most students, observing how mathematicians work when they do research will not be a feasible way to gain insights into inquiry processes in mathematical research.

However, such inquiry processes can be made visible and accessible for students to experience by studying the history of the subject matter of mathematics. In such investigations students can examine processes of how mathematicians have generated the ideas, constructed the concepts and produced the mathematics they read about in their textbooks. But, again, we have to keep in mind that the mere exposure to historical development processes of mathematics is not enough for students to develop informed conceptions about the nature of mathematics – for this to happen, students must be challenged to reflect explicitly and critically upon concrete aspects of the nature of mathematics like the ones listed above.

3. PPL – an explicit-reflective framework for teaching about the nature of mathematics in an inquiry learning environment

PPL stands for Problem-oriented Project Learning and is short for the pivotal pedagogical principle underlying the organization of the all study programs at Roskilde University (RUC), Denmark. It refers to the principle of *problem-oriented participant-directed project work* as it has been developed and is practiced at RUC.[4] Common for all programs is that in each semester the students use half of their time on regular course work. The other half of their study time they work in groups of two to eight students on a project which objective is to solve (or analyze or formulate or make solvable) a problem of the group's own choice under supervision of a professor. The students work on the two strands (courses and project work) of study activities simultaneously – however, there is a so-called 'project-intensive' period of 1½-3 weeks each semester, depending on the specific study program.

The project work is documented throughout the semester through outlines, work- and discussion-papers written by the students. These are discussed and planned for at weekly meetings with the supervisor. At the end, the group writes a coherent report of 50-100 pages, in which the students state and argue for their problem and its relevance, their methodology and its validity, their choice of theories, experiments, data-collections, analyses, results and conclusions. They make a critical evaluation of their solution. The students consult textbooks, research literature and experts in their field(s) of investigation. During the project work, they often divide the jobs between them in various ways, but in the end every student in a group is responsible for the entire project and its report. The project work is evaluated at an oral group-examination with an external examiner. The students are responsible for every aspect of the project work: the formulation of their problem, the argumentation for its relevance, formulation and justification of hypotheses, the choice of methodology and discussions of its strengths and weaknesses. They are supposed to be able to reflect explicitly and concretely about how their solution/analysis depends on their methodology. They are held accountable for the design and performing of experiments and/or analyses based on theories and the construction of models. The students are in charge of the project work from beginning to end, supported by various milestones throughout the semester.

One of the projects the students work on at the master level in mathematics is constrained by the theme: 'mathematics as a scientific discipline'. In the study regulation, this type of project is described in the following way:

> The project should deal with the nature of mathematics and its "architecture" as a scientific subject such as its concepts, methods, theories, foundation etc., in such a way that the nature of mathematics, its epistemological status, its historical development and/or its place in society gets illuminated.

Every mathematics student at RUC at the master level will work on a project that fulfills these requirements. They will do so in a group with other students. Each group will work on their own problem as described above, but common for all the problems is that the students, through their work within a group, gain experiences with mathematics as a scientific subject as described in the quote above from the study regulation. Hence, there are explicit references to the nature of mathematics learning objectives for the students within the curriculum of the master's program in mathematics.

The problem orientation of the project work together with the study regulation of the 'mathematics as a scientific subject' guaranties that the students work with a specific problem that address (aspects of) the nature of mathematics, in such a way that it is anchored in the subject matter of mathematics (its concepts, methods, theories, foundation etc.). Through this anchoring, the students' reflections become contextualized and concretized. That is, the problem orientation together with the curriculum description of the 'mathematics as a scientific subject'-project provides structured opportunities for the students to "examine their science [mathematics] learning experiences from within an epistemological framework" as required by Abd-El-Khalick (2013, 2091) in order to have a reflective framework. The students gain their mathematics as a science learning experience from the concrete attachment of their problem within the subject matter of mathematics. The aspects of the nature of mathematics that their problem is addressing provide the epistemological context in which the students examine their mathematics-as-a-science learning.

So argued, the problem-oriented project work under the theme 'mathematics as a scientific subject' as it is carried out in the PPL-model at Roskilde University, provides an explicit-reflective framework for teaching about the nature of mathematics in the inquiry pedagogy defined by the principles and the organization of the project work at RUC. Be aware, that inquiry pedagogy refers to the study and learning environment created by the organization of the problem-oriented project work most notably with the students' autonomy throughout the whole process of the project work.

The problem-orientation together with the description of the 'mathematics as a science subject'-project also establishes a history, philosophy and/or sociological inquiry learning environment depending on the students' choice of problem and methodology.

In the next section we will we see how students within PPL under the theme 'mathematics as a scientific subject' in the master's program in mathematics have opportunities

1) to also work 'mathematical inquiry', that is, to work in a way that resemblances how mathematical research is done,
2) to develop informed conception of the generation and validation of mathematical knowledge.

4. Students' Development of Informed Conception of Mathematical Knowledge Production

In this section two examples of PPL-projects of the 'mathematics as a scientific subject' type will be presented and discussed to illustrate how the students in the respective project-groups, through history, developed informed conception about the epistemology of mathematics, about how mathematicians produce mathematical knowledge, what kind of questions that drive mathematical research and how mathematical research is validated, while experiencing a mathematical inquiry learning environment that gave them insight into authentic mathematical practices.

The first project report[5] has the title *Generalizations in the Theory of Integration: An Investigation of the Lebesgue Integral, the Radon Integral and the Perron Integral*. It is a 75 page report written by two students who were curious about the "need" in mathematics for further integrals than the Riemann integral. In the textbook of their first analysis course they had stumbled over a footnote in which it was pointed out that there are other types of integrals e.g., the Lebesgue integral. The students soon discovered that there are many other integrals e.g., the Denjoy, the Perron, the Henstock, the Radon, the Stieltjes and the Burkill integral, as they wrote in their report[6]:

> All these integrals are most often described in the literature as *generalizations*, and sometimes as *extensions*, of either the Riemann or the Lebegues integral. This gave rise to questions such as: What do these integrals do? Why have so many types of integrals been developed? Why is it always the Lebesgue integral we hear about? What is meant by generalizations in this respect? In what sense are the various integrals generalizations of former definitions of integrals? Are the generalizations of the same character? (Timmermann and Uhre, 2001, p. 1, italic in the original)

And they continue in the next paragraph with their own motivation:

> the above mentioned questions are related to an overall wondering of ours about *how* mathematics develops and *what* it is that drives this development, i.e., *why* mathematics develops as it does. (Timmermann and Uhre, 2001, p. 1, italic in the original)

The students chose to investigate this by looking into the history of mathematics, as they explained in their report:

> The development of mathematics depends on the people who create the new mathematics and the importance of the mathematics that is being developed. In order for new mathematics to be created within an area of mathematics, there have to be some mathematicians who are actually interested in investigating and uncovering this area. By studying the motivation of the people who have helped to develop the integral concepts, we can get an insight into why a mathematical area is being studied. (Timmermann and Uhre, 2001, p. 2)

Supported by literature from historians of mathematics, the students traced and read mathematical papers and books of Lebesgue, Perron and Radon in which these mathematicians developed – or worked with mathematical ideas that motivated them to develop – the integrals that bear their names.[7] The students analyzed and compared the sources with respect to the motivation of the mathematicians, why they created these generalized integral concepts, the differences and similarities between the characteristics and scope of the generalizations, guided by the following research questions that the students had formulated for their project work:

> What motivated Lebesgue, Perron and Radon in their pursuit of their generalizations of the integral?
>
> What are the characteristics and scope of the generalizations by Lebesgue, Perron and Radon, and what are the differences among them?
> (Timmermann and Uhre 2001, p. 3)

On the one hand, by focusing on understanding the mathematicians' motivation for generalizing the integral concept when reading the historical sources, the students became engaged with mathematical inquiry analogue to some research processes in mathematics as they are carried out by working mathematicians. The students work was guided by historical and philosophical questions, but they answered these questions through analyses of the mathematical content, definitions, theorems, proofs and techniques that were stated, treated and worked out in the sources. Through their analyses of the sources, the students gained first-hand experiences with research processes in the production of mathematical knowledge by studying the masters, so to speak.

On the other hand, by focusing on analyzing and identifying the characteristics of the various generalizations of the integral concept, the students came to reflect upon their inquiry investigations from within an epistemological framework. This last question structured the students' critical reflections about the function, assessment and significance of the various generalizations of the integral concept.

The second project report has the title: *The Real Numbers – Constructions in the 1870s*. The project was carried out by a group of six students. They wrote a report of 56 pages in which they answered the following question:

> Why did a need for a construction of the real numbers emerge around the 1870s?

These students were puzzled when they realized that mathematicians had worked with the real numbers on an intuitive foundation for years and years before it (the foundation of the real numbers) 'all of a sudden' became a problem that needed to be solved. The students explained their own motivation in their report as follows:

> ... it [the mathematics teaching we were exposed to in primary, and in lower and upper secondary school] does not provide any kind of understanding of where mathematics comes from, which people were involved, who defined or constructed the concepts and tools we use today. Well, of course, most of us will become acquainted with some key persons from the history of mathematics, for example Newton and Euclid, but the fewest will obtain a proper overview. Who knows, for example, how the real numbers emerged or were constructed? And especially: who knows that the real numbers had foundational problems and only then were constructed?
>
> Basically we had the idea that one mathematician saw that it was a problem that a construction of the real numbers was lacking, solved it and then presented it [the construction of the real numbers] as a solved problem. However, we quickly discovered that this was not the case. There was not just one mathematician who believed that there was a

> problem, but several. This indicated that the lack of a construction of the real numbers had become a problem in connection with developments of mathematics up to then. (Wandahl et. al., 2004, p. 1-2)[8]

The students' motivation and their research question (the problem that guided their project work) are concerned with a fundamental aspect of the nature of mathematics as a scientific subject, namely what do mathematicians wonder about? How do the problems that mathematicians struggle to solve emerge? Where do these problems come from? How are they connected with ongoing research in mathematics? When, why and by whom are they deemed so important, that they become essential problems of the field in need of a solution? These questions structured the students' analyses of the mathematical sources and their reflections of how mathematical knowledge was generated in this particular episode in the history of mathematics.

To place the discussion in the historical context, the students studied the paper *RienanalytischerBeweis des Lehrsatzes, dasszwischen je zweyWerthen, die einentgegensetztesResultatgewären, wenigstgenseinereelleWurzel der Gleicheung liege* by Bernard Bolzano (1817) in which he proved the mean value theorem, and discussed methodological and epistemological issues in mathematics. The students considered Blozano's work as an indication that such concerns were present in mathematical research during the nineteenth century. In order to answer their own research questions for the project work, the students chose to investigate the work of Dedekind and Cantor. They realized that these two mathematicians had different reasons for constructing the real numbers with Dedekind's motivation growing out of a concern for teaching, whereas Cantor ran into the problem of the construction of the real numbers in his work with trigonometric series. Based on the work of historians of mathematics, the students discussed the various conceptions of the arithmetization of analysis in the 1870s, in order to place their analyses of the historical sources in a broader historical context. The students became aware that mathematicians do not always agree. They quoted Frege's critic of the constructions of the real numbers:

> How could one be sure, in all these constructions, that the new concepts or rules introduced did not lead to contradictions, violating the minimal criterion accepted by all parties for the possibility of a mathematical concept. (Wandahl et. al., 2004, p. 42 [quoted from Epple (2003), p. 302)]

As was the case with the project on generalizations of the integral concept, the students were invited into the mathematical 'workbench' of Bolzano, Dedekind and Cantor among others. Through their work with the historical sources, the students became engaged with mathematical inquiry. The requirement for the "mathematics as a scientific subject"-project and their specific problem-orientation provided the students with an explicit-reflective framework that gave opportunities for them to reflect upon their inquiry investigations of the mathematical content of the sources from within an epistemological framework of how this particular mathematical knowledge was generated, validated and discussed among mathematicians at the time.

The two project reports illustrated how the students acquired informed conceptions about the nature of mathematics coming from different perspectives: In the first project, the students obtained concrete experiences with processes of forming new objects in mathematics with reference to already existent objects[9], through generalizations – and how different processes of generalization can be distinguished with respect to whether the new object is an abstraction of the already existent one or whether the new object is an extension in the sense, that is contains the object it is a generalization of. In the second project, the students experienced how problems of foundational and epistemological nature arose with respect to the foundation of the real numbers, how this generated new research in mathematics and gave rise to discussions and debates among mathematicians – aspects which are important parts of mathematical inquiry, but rarely occur in traditional mathematics teaching. However, such aspects can be brought forward and made explicit objects of students' reflection through history of mathematics.

Characteristics for both projects are that the students work was embedded in rich and mathematically thick episodes from the history of mathematics about motivations and driving forces behind the development of mathematics, that the students became immersedin mathematical inquiry analogue to how mathematicians do research in 'pure' mathematics, and that they became engaged with structured and critical reflections about these aspects within an epistemological framework.

5. Teaching withthe nature of mathematics through history in an inquiry learning environment

The nature of mathematics (NOM) is explicit in the mathematics curriculum for upper-secondary school in Denmark. One of the learning goals for mathematics is that the students should be able to demonstrate knowledge of how mathematics has developed in interaction with the historical, scientific and cultural development. Another learning goal is that students should demonstrate knowledge of the identity and methods of mathematics.

In this section we shall see an example of how a mathematics (teacher) student from Roskilde University used her knowledge about the nature of mathematics that was formed within the explicit-reflective framework which was unfolded and illustrated above, to teach with and about the nature of mathematics in a Danish high school. She used her knowledge and experiences from her own mathematics education to enact an inquiry learning environment that invited her 17 year old high school students (11^{th} graders) into inquiry processes that bore some resemblance with authentic mathematical practices. She created the learning environment by having the students read excerpt from original historical sources from the development of the function concept i.e., by having the students studying the masters. By setting up explicit learning goals for the students that addressed historical and mathematical questions related to the development of the function concept, and by having the students analyze the original sources to answer these questions, the teacher designed a learning environment in which she taught both *with* and *about*the nature of mathematics. Altogether she spent 13 lessons of 50 minutes each with the students in the classroom – and the students were expected to spend an equal amount or more of time working on the various tasks at home.

The intentions for the students' learning outcome reflect the explicit-reflective framework (Petersen, 2011):

- The students will acquire an understanding of what is involved in the modern definition of a function
- The students will come to reflect upon the concept of a function – what is a function?
- The students will acquire an understanding of our modern function concept as a result of a historical developmental process
- The students will gain insights into the role played by the human actors [former mathematicians] in the development of the function concept.

The teaching module was designed in two steps: In step 1 the students were divided into four so-called basic-groups that had specific but different tasks. Group 1 worked with various historical definitions of a function, group 2 worked with the debate of the motion of a vibrating string primarily between the Swiss mathematician Leonhard Euler (1707-1783) and the French mathematician Jean le Rond D'Alembert (1717-1783), group 3 situated the main actors (Euler and the German mathematician Peter G. L. Dirichlet (1805-1859)) in their respective societies and the time in which they lived, and group 4 worked on the modern concept of a function. Each group received a working sheet from the teacher with explicit requirements for their work. Group 1, e.g.,was given Danish translations of an extract from Eulers' (1748) book *Introductio in AnalysinInfinitorum*with the definition of a function, and an explanation of how Euler later in 1748 extended his original function concept, and an extract from Dirichlet's paper (1837) Über die DarstellungganzwilkürlicherFunctionendurch Sinus- und Cosinusreihen. Based on these three texts the students received the following task:

> Explain what a function is according to Euler's original definition of a function in Introductio in AnalysinInfinitorum, Euler's extended definition of a function and Dirichlet's definition of a function. Describe how these three definitions of a function are different from each other and in what

> ways they are similar. Explain what the principle of the generality of the variable is all about and the relationship between this principle and the principle of the generality of the validity of analysis. (Petersen 2011, Appendix B)

The task was followed by eight questions which were meant both as a help for the students to 'de-code' the task and to make the requirements and the expectations of the teacher for the students' work more explicit, and to qualify the students' reflections and structure their work:

> (1) What are the central concepts in Euler's definition of a concept? (2) Which principle characterizes a variable according to Euler, and what is this principle called? (3) What is the principle of the generality of the variable all about? (4) What are the similarities between the principle of the generality of the variable and the principle of the generality of the validity of analysis? Consider why both principles have been given names that contain the word "general".[10] (5) How does Euler's extended concept of a function differ from his original concept, and what are the similarities? (6) Find three ways in which Dirichlet's concept of a function differs from Euler's definition. (7) Explain from text 3, what Dirichlet must have thought about the generality of the variable. (8) On page 10 there are four pictures. Which of these pictures are graphs of functions according to Euler's definition in *Introductio in AnalysinInfinitorum*, Euler's extended definition and Dirichlet's definition respectively? (Petersen 2011, Appendix B)

In step 2 new groups were formed in such a way that each group had at least one participant from each of the four basic-groups. This meant, in least in principle, that all the knowledge that had been developed in the basic groups was present in each of the new groups. In contrast to the basic-groups, the new groups, also called the expert-groups, all worked on the same assignment. They were asked to write an essay about a fictional debate between mathematicians, where one part is claiming that mathematical concepts are static, timeless entities that exist independent of humans and society. In opposition to this, the other part reinforce that mathematical concepts develop over time, that they are the results of research processes. The students received a made-up invitation from the journal NORMAT to contribute to this debate by submitting a paper for the journal on this issue. The students should argue for their own opinions based on the collected work that had been done in the basis-groups. The teacher had formulated four issues that the students had to address and discuss in their paper: Euler's, Dirichlet'sand our current concept of a function; the two meta-rules i.e., the generality of the variable and the general validity of analysis, the raison d'etre behind domain, range of image and proofs; sociological factors that had influenced the development of the function concept; and human factors.

The analysis of data that was collected during the teaching shows that the high school students realized that the concept of a function, they read about in their textbook, was the result of a historical development, and they also gained more specific knowledge about some of the key elements of this development. The students became immersed in mathematical inquiry processes by tracing (parts of) the path of the masters, as is illustrated by the following quotes from one of the essays written in the expert-groups. The quotes show that these students became aware of at least one source for mathematical research questions as well as of discussions among mathematicians that relate to the validation of mathematical knowledge:

> The reason why Euler began to work with discontinuous functions was because of a debate between contemporary mathematicians. The debate concerned the fact that the functions the mathematicians worked with could not describe a vibrating string.

> [...] the development of the concept of a function was among other things due to human attitudes and interpretations, which were important factors. For example, some of Euler's contemporary mathematics colleagues were of the opinion that Euler's extended function concept should not be used because it went against the principle of mathematics. They thought it was cheating. This meant that Euler's extended function concept never came to be used as intended, and a new function concept was developed by Dirichlet.

We will not go further into the results regarding the learning of the high school students, for this we refer to Kjeldsen and Petersen(2014). Here we will restrict ourselves to pointing out that the teacher's knowledge *about* the nature of mathematics and her informed epistemological conception about the production and validation of mathematical knowledge (which she had obtained within the explicit-reflective framework of the 'mathematics as a scientific discipline'-project during her mathematics studies at Roskilde University), enabled her to design and implement the course described above in high school. A course where she taught both *with* and *about* the nature of mathematics through the history of mathematics; i.e., she taught in a way that immersed the high school students in mathematical inquiry processes and made them capable of forming epistemological understandings of how mathematical knowledge is generated and validated.

6. Conclusion

The problem-oriented project work in the master's program in mathematics at Roskilde University provides and explicit-reflective framework in the sense ofAbd-El-Khalick (2013). The analyses of the two student projects on generalizations in the theory of integration and the construction of the real numbers respectively, demonstrate how the students in this program through working with rich and mathematically thick episodes from the history of mathematics, can develop informed conception about the epistemology of mathematics, of how mathematicians produce mathematical knowledge, what kind of questions that drive mathematical research, and how mathematical research is validated, while experiencing an inquiry learning environment that gives them insight into authentic mathematical practice. In the course of their problem-oriented project work they came to reflect upon and criticize the way mathematicians generate and validate mathematical knowledge, i.e., inquiry processes in mathematical research were made explicit objects for the students' reflections within an epistemological framework.

The analysis of the design and implementation of the teaching in high school of the historical development of the function concept exemplifies how the RUC master program in mathematics enables its graduates to use their integrated understanding of history and philosophy of mathematics to teach both *with* and *about* the nature of mathematics in the sense of Abd-El-Khalick (2013) by endorsing learning environments that, to the extent explored and in the sense described above, bear resemblance to authentic mathematical practices, that is, to establish mathematical inquiry learning environments through history.

Notes

1. In the Rocard report, the term science also includes mathematics.
2. In Denmark, upper secondary mathematics teachers have a university degree in mathematics. They become high school teachers in mathematics by applying for a teaching job at a school after they graduated from the university. During the first year or two as a teacher, they go through a pedagogical and mathematics didactics education.
3. Research from science education has documented that "while inquiry might serve as an ideal context for helping students and teachers develop informed NOS [nature of science] views, it does not follow that engagement with inquiry would necessarily result in improved understandings." (Abd-El-Khalick 2013, p. 2089). For further references with documentation see Abd-El-Khalick (2013).
4. For a thorough description of the pedagogical foundation of PPL at Roskilde University and the various student-centered learning concepts that are combined in the Roskilde Model of problem-oriented learning and project work see Andersen and Heilesen (2015).
5. The students' project report can be downloaded at the following address:
   http://milne.ruc.dk/ImfufaTekster/pdf/403.pdf

6. The translations into English have been done by the author.
7. For example, with respect to Lebesgue, the students read hisnote *Sur unegénéralisation de l'intégraledéfinie*which was published in *ComptesRendus de l'Académie des Sciences de Paris* in 1901 and his thesis *Intégrale,Longueur, Aire*from 1902.
8. The translations into English have been done by the author.
9. See Kjeldsen and Carter (2012) for a philosophical discussion of the growth of mathematical knowledge with a special focus on the question of how mathematical objects are introduced to mathematical practice.
10. These questions address the issue of what Sfard (2008) calls meta-discursive rules in mathematics. For a discussion of these, see Kjeldsen and Blomh j (2012) and Kjeldsen and Petersen (2014).

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